

Intersection numbers of spectral curves

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Abstract

We compute the symplectic invariants of an arbitrary spectral curve with only 1 branchpoint in terms of integrals of characteristic classes in the moduli space of curves. Our formula associates to any spectral curve, a characteristic class, which is determined by the laplace transform of the spectral curve. This is a hint to the key role of Laplace transform in mirror symmetry. When the spectral curve is $y = \sqrt{x}$, the formula gives Kontsevich–Witten intersection numbers, when the spectral curve is chosen to be the Lambert function $e^x = ye^{-y}$, the formula gives the ELSV formula for

Hurwitz numbers, and when one chooses the mirror of \mathbb{C}^3 with framing f , i.e. $e^{-x} = e^{-yf}(1 - e^{-y})$, the formula gives the topological vertex Mariño–Vafa formula, i.e. the generating function of Gromov-Witten invariants of \mathbb{C}^3 . In some sense this formula generalizes ELSV, Mariño–Vafa formula, and Mumford formula.

1 Introduction

In [16] were introduced some "symplectic invariants" of a spectral curve (we call spectral curve a plane curve, i.e. a Riemann surface embedded into \mathbb{C}^2 , often chosen as the locus of zeroes of an analytical function $E(x, y) = 0$). Those invariants play an important role in random matrix theory, and in many enumerative geometry problems. They were first introduced in relationship with random matrices and enumeration of discrete

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surfaces. Indeed, if the spectral curve \mathcal{S} is chosen as the spectral curve of a matrix model, then the g^{th} symplectic invariant $F_g(\mathcal{S})$ is the g^{th} term in the large size expansion of the matrix integral, and it is the generating function enumerating discrete surfaces of genus g (in fact this is the property which initially motivated the definition of symplectic invariants [12]).

Later it was realized that symplectic invariants of the spectral curve $y^2 = x$ are the generating function of intersection numbers of Chern classes in $\mathcal{M}_{g,n}$ (the moduli space of curves of genus g with n marked points), i.e. Witten-Kontsevich intersection numbers [13].

Then it was realized that they can also encode Gromov-Witten invariants [28, 7]. If \mathfrak{X} is a toric Calabi-Yau 3-fold, and $\mathcal{S} = \hat{\mathfrak{X}}$ is its mirror singular curve (the mirror [19] of a toric CY3, is a CY3 whose singular locus is a plane curve, which we call the mirror curve $\hat{\mathfrak{X}}$), then it was conjectured by Mariño [28] and then more precisely by Bouchard-Mariño-Klemm-Pasquetti in [7], that $F_g(\hat{\mathfrak{X}})$ is the generating function of Gromov-Witten invariants counting stable maps of genus g into \mathfrak{X} (BKMP conjecture [7]). That conjecture was proved in a few cases [9, 37].

So we see that for "good" choices of spectral curves, the symplectic invariants have a beautiful enumerative geometry interpretation, they count some "number" of surfaces, or some "intersection numbers" in the moduli space of curves or maps.

However, for an arbitrary spectral curve, different from the "good ones" listed above, it was so far not really known what symplectic invariants were counting.

Here in this article, we relate the symplectic invariants of an arbitrary spectral curve, to some intersection numbers. Our formula is reminiscent of the ELSV formula [10, 11] (relating Hurwitz numbers to intersection numbers of one Hodge class), or Mariño-Vafa formula [29] for the vertex case, and the Mumford formula [32] (which rewrites the Hodge class in terms of ψ and κ classes).

Our formula for only one branchpoint is given by theorem 3.3 below, which we write here:

Theorem 3.3

$$W_n^{(g)}(\mathcal{S}_a; z_1, \dots, z_n) = 2^{d_{g,n}} \sum_{d_1 + \dots + d_n \leq d_{g,n}} \prod_i d\xi_{d_i}(z_i) \left\langle e^{\frac{1}{2} \sum_\delta \iota_{\delta*} \hat{B}(\psi, \psi')} e^{\sum_k \tilde{t}_k \kappa_k} \prod_i \psi_i^{d_i} \right\rangle_{g,n} \quad (1.1)$$

where the notations will be defined below. To stay in an introductory level, we just point out that the left hand side of that formula is defined from the geometry of a spectral curve, i.e. a complex plane curve (therefore a type B quantity), whereas the right hand side contains intersection numbers of homology classes in some moduli-space, i.e. a type A quantity. This looks like a kind of mirror symmetry [19], where

the "mirror map" relates the B-moduli of the spectral curve, to the type A moduli \tilde{t}_k , $d\xi_d$, \hat{B} in the right hand side, by Laplace transform.

Let us now define our notations.

2 Symplectic invariants of spectral curves

2.1 Spectral curves

Intuitively, a spectral curve \mathcal{S} is a plane curve, i.e. the locus of solutions of an analytical function $E(x, y) = 0$ in \mathbb{C}^2 . In particular, it defines a Riemann surface \mathcal{C} , and two projections $x : \mathcal{C} \rightarrow \mathbb{C}$ and $y : \mathcal{C} \rightarrow \mathbb{C}$. By definition $\{(x, y) \mid E(x, y) = 0\} = \{(x(z), y(z)) \mid z \in \mathcal{C}\}$. For example for the spectral curve $y^2 = x$, we have $\{(x, y) \mid y^2 = x\} = \{(z^2, z) \mid z \in \mathbb{C}\}$.

For our purposes, it is more convenient to define the spectral curve directly from the parametrization (\mathcal{C}, x, y) where \mathcal{C} is a complex curve (a Riemann surface, not necessarily compact), and x and y are two analytical functions $\mathcal{C} \rightarrow \mathbb{C}$.

Definition 2.1 *A spectral curve \mathcal{S} , is the data of:*

$$\mathcal{S} = (\mathcal{C}, x, y, B) \tag{2.1}$$

- \mathcal{C} is a complex curve (a Riemann surface, not necessarily compact),
- x and y are two analytical functions $\mathcal{C} \rightarrow \mathbb{C}$.
- $B(z, z')$ is a "Bergman kernel", i.e. a symmetric 2nd kind differential on $\mathcal{C} \times \mathcal{C}$, having a double pole at $z = z'$ and no other pole, and which behaves like

$$B(z, z') \underset{z \rightarrow z'}{\sim} \frac{dz \otimes dz'}{(z - z')^2} + O(1) \tag{2.2}$$

in any local parameter z . (In general the Bergman kernel is not unique, one may add to it anything which is holomorphic (with no pole) in $\mathcal{C} \times \mathcal{C}$ and symmetric).

- An important example of spectral curve is:

$$y^2 = x \tag{2.3}$$

which can be parametrized by two functions $x(z) = z^2$ and $y(z) = z$ for a complex variable $z \in \mathcal{C} = \mathbb{C}$, and with the Bergman kernel

$$B(z, z') = \frac{dz \otimes dz'}{(z - z')^2}. \tag{2.4}$$

In other words

$$\mathcal{S} = (\mathbb{C}, x(z) = z^2, y(z) = z, B). \quad (2.5)$$

This spectral curve, often called "Airy curve"² plays an important role in Witten–Kontsevich theory, as we shall see below.

- Another interesting example is:

$$e^x = y e^{-y}, \quad (2.6)$$

called the "Lambert curve", because $y = L(e^x)$ where L is the Lambert function. It can be parametrized by $x(z) = -z + \ln z$, $y(z) = z$, $z \in \mathcal{C} = \mathbb{C}^* \setminus \mathbb{R}_-$, and the Bergman kernel is again chosen to be

$$B(z, z') = \frac{dz \otimes dz'}{(z - z')^2}. \quad (2.7)$$

Our spectral curve is thus

$$\mathcal{S} = (\mathbb{C}^* \setminus \mathbb{R}_-, x(z) = -z + \ln z, y(z) = z, B). \quad (2.8)$$

This spectral curve plays an important role in Hurwitz numbers counting (as was noticed by Bouchard and Mariño [8] and proved in [15, 6]), and reproved below in section 8.2 as a consequence of theorem 3.3.

- Another interesting example is:

$$e^{-x} = e^{-fy} (1 - e^{-y}), \quad (2.9)$$

called the "topological vertex curve", because it is the mirror curve of the framed topological vertex ($f \in \mathbb{Z}$ is the framing), indeed writing $X = e^{-x}$ and $Y = e^{-y}$, it satisfies:

$$X = Y^f (1 - Y). \quad (2.10)$$

It can be parametrized by $x(z) = -f \ln z - \ln(1 - z)$, $y(z) = -\ln z$, $z \in \mathcal{C} = \mathbb{C}^* \setminus (-\infty, 0] \cup [1, \infty)$, and the Bergman kernel is again chosen to be

$$B(z, z') = \frac{dz \otimes dz'}{(z - z')^2}. \quad (2.11)$$

Our spectral curve is thus

$$\mathcal{S} = (\mathbb{C}^* \setminus (-\infty, 0] \cup [1, \infty), x(z) = -f \ln z - \ln(1 - z), y(z) = -\ln z, B), \quad (2.12)$$

This spectral curve plays an important role in the Gromov-Witten theory of \mathbb{C}^3 as was noticed by [7] and proved in [9, 37], and reproved below in section 4 as a consequence of theorem 3.3.

²It is often called Airy curve because its symplectic invariants can be written in terms of Airy function. For instance $\sum_g W_1^{(g)}(z) = (Ai'(z^2)Bi'(z^2) - z^2 Ai(z^2)Bi(z^2))2zdz = -\sum_g (6g - 3)!!/(g! 3^g 2^{5g-1}) z^{2-6g} dz$.

2.2 Branchpoints

A branchpoint is a zero of dx , i.e. $dx(a) = 0$. Let us assume that a is a regular branchpoint, i.e. it is a simple zero of dx , and we have $dy(a) \neq 0$. This means that locally near a the curve has a square-root behavior:

$$y(z) \sim y(a) + y'(a) \sqrt{x(z) - x(a)} + O(x(z) - x(a)) \quad (2.13)$$

or also, that $\zeta(z) = \sqrt{x(z) - x(a)}$ is a good local parameter near a .

In the vicinity of a , the square root has two branches, and we denote \bar{z} the point corresponding to the other sign of the square-root, i.e. the unique other point in the vicinity of a such that $\zeta(\bar{z}) = -\zeta(z)$, or also

$$x(\bar{z}) = x(z). \quad (2.14)$$

Notice that \bar{z} is well defined only when z lies in the vicinity of a branchpoint, it is (in general) not globally defined for any $z \in \mathcal{C}$.

Near a branchpoint a , we can Taylor expand, and define the "Kontsevich times":

Definition 2.2 *The times t_k of a branchpoint a , are the coefficients of the Taylor series:*

$$y(z) \sim \sum_{k=0}^{\infty} t_{k+2} \zeta^k \quad , \quad \zeta = \sqrt{x(z) - x(a)}. \quad (2.15)$$

- Example for the Airy spectral curve $\mathcal{S} = (\mathbb{C}, x(z) = z^2, y(z) = z, B)$, we have $dx(z) = 2zdz$, which vanishes at $z = a = 0$, there is only one branchpoint. We clearly have $\bar{z} = -z$, which in this case is globally defined. In that case, we have the times:

$$t_k = \delta_{k,3}. \quad (2.16)$$

- Example for the Lambert spectral curve $\mathcal{S} = (\mathbb{C}^* \setminus \mathbb{R}_-, x(z) = -z + \ln z, y(z) = z, B)$, we have $dx(z) = \frac{1-z}{z} dz$, which vanishes at $z = a = 1$, there is only one branchpoint. Near $z = 1$ we have $\bar{z} = 2 - z + \frac{2}{3}(z-1)^2 + \dots O(z-1)^3$, which in this case is not globally defined. In that case the times t_k are given by

$$y = 1 + i\sqrt{2}\zeta - \frac{2}{3}\zeta^2 + \frac{11i\sqrt{2}}{9}\zeta^3 + \dots \quad , \quad \zeta = \sqrt{x+1}. \quad (2.17)$$

2.3 Symplectic invariants

Symplectic invariants were introduced in [16]. For a spectral curve $\mathcal{S} = (\mathcal{C}, x, y, B)$, we define:

Definition 2.3 The "symplectic invariant descendents" $W_n^{(g)}(\mathcal{S}; z_1, \dots, z_n)$ with $n \geq 1$ and $g \geq 0$ are defined by:

$$W_1^{(0)}(z) = y(z) dx(z), \quad (2.18)$$

$$W_2^{(0)}(z, z') = B(z, z'), \quad (2.19)$$

and for $2g - 2 + n > 0$, by the "topological recursion" [12, 2]

$$\begin{aligned} W_{n+1}^{(g)}(z_0, \overbrace{z_1, \dots, z_n}^J) &= \sum_{a=\text{branch points}} \text{Res}_{z \rightarrow a} K(z_0, z) \left[W_{n+2}^{(g-1)}(z, \bar{z}, J) \right. \\ &\quad \left. + \sum_{h=0}^g \sum_{I \subset J}' W_{1+\#I}^{(h)}(z, I) W_{1+n-\#I}^{(g-h)}(\bar{z}, J \setminus I) \right] \end{aligned} \quad (2.20)$$

where \sum' means that we exclude from the sum all terms which contain a factor $W_1^{(0)}$, and the recursion kernel K is:

$$K(z_0, z) = \frac{\int_{z'=\bar{z}}^z B(z, z')}{2(y(z) - y(\bar{z})) dx(z)}. \quad (2.21)$$

$W_n^{(g)}(z_1, \dots, z_n)$ is a meromorphic 1-form for each $z_i \in \mathcal{C}$, it is symmetric in all the z_i 's. For $2g - 2 + n > 0$, it has poles only at branchpoints, without residues, and the degree of the poles are $\leq 6g - 4 + 2n$.

$W_n^{(g)}$ with $2 - 2g - n < 0$ are called stable, and those with $2 - 2g - n \geq 0$ are called unstable (only $W_1^{(0)}$ and $W_2^{(0)}$ are unstable).

The symplectic invariants themselves are $F_g = W_{n=0}^{(g)}$ for $n = 0$, and are defined as follows

Definition 2.4 For $g \geq 2$, the symplectic invariants of \mathcal{S} are defined by

$$F_g(\mathcal{S}) = W_0^{(g)}(\mathcal{S}) = \frac{1}{2-2g} \sum_{a=\text{branch points}} \text{Res}_{z \rightarrow a} W_1^{(g)}(z) \Phi(z) \quad (2.22)$$

where $\Phi(z)$ is any function defined locally near a such that

$$d\Phi = y dx. \quad (2.23)$$

For $g = 1$, F_1 is defined as

$$F_1(\mathcal{S}) = \frac{1}{24} \ln \left(\tau_B(\{x_i = x(a_i) \mid a_i = \text{branch points}\}) \prod_{a=\text{branch points}} y'(a) \right) \quad (2.24)$$

where $y'(a) = \lim_{z \rightarrow a} \frac{y(z) - y(a)}{\sqrt{x(z) - x(a)}}$, and $\tau_B(x_1, \dots, x_k)$ is the Bergman Tau-function defined by [22]

$$\frac{\partial \tau_B(x_1, \dots, x_k)}{\partial x_i} = \text{Res}_{z \rightarrow a_i} \frac{B(z, \bar{z})}{dx(z)} \quad (2.25)$$

where $x_i = x(a_i)$ are the x -projections of branchpoints. There is also a definition of $F_0(\mathcal{S})$, see [16], but we shall not use it in this article, we just notice that $F_0(\mathcal{S})$ doesn't depend on the Bergman kernel.

Our goal, is to relate symplectic invariants of an arbitrary spectral curve, to the combinatorics of intersection numbers in the moduli space of curves.

2.3.1 Examples with 1 branchpoint

Assume that there is only one branchpoint at $z = a$. It is convenient to define:

$$d\xi_d(z) = - \operatorname{Res}_{z' \rightarrow a} B(z, z') \frac{(2d-1)!!}{2^d (x(z') - x(a))^{d+\frac{1}{2}}}$$

and the Taylor expansion of $y(z)$ near $z = a$:

$$\begin{aligned} y(z) &= y(a) + \sum_{k=0}^{\infty} t_{k+2} (x(z) - x(a))^{\frac{k}{2}} \\ &= y(a) + t_3 (x(z) - x(a))^{\frac{1}{2}} + t_4 (x(z) - x(a)) + t_5 (x(z) - x(a))^{\frac{3}{2}} + \dots \end{aligned}$$

and the Taylor expansion of $B(z, z')$ near a :

$$\begin{aligned} B(z, z') &= \frac{dx(z) \otimes dx(z')}{4\sqrt{x(z) - x(a)} \sqrt{x(z') - x(a)}} \left[\frac{1}{(\sqrt{x(z) - x(a)} - \sqrt{x(z') - x(a)})^2} \right. \\ &\quad \left. + \sum_{k,l} B_{k,l} (x(z) - x(a))^{\frac{k}{2}} (x(z') - x(a))^{\frac{l}{2}} dx(z) dx(z') \right] \end{aligned}$$

For low values of g and n , a direct computation of residues gives:

•

$$W_3^{(0)}(z_1, z_2, z_3) = \frac{1}{2t_3} d\xi_0(z_1) d\xi_0(z_2) d\xi_0(z_3) \quad (2.26)$$

•

$$W_1^{(1)}(z) = \frac{1}{24t_3} \left(d\xi_1(z) - \frac{3t_5}{2t_3} d\xi_0(z) \right) + \frac{B_{0,0}}{4t_3} d\xi_0(z) \quad (2.27)$$

•

$$\begin{aligned} W_4^{(0)}(z_1, z_2, z_3, z_4) &= \frac{1}{2t_3^2} (d\xi_1(z_1)d\xi_0(z_2)d\xi_0(z_3)d\xi_0(z_4) + \text{sym}) \\ &\quad - \frac{3t_5}{4t_3^3} d\xi_0(z_1)d\xi_0(z_2)d\xi_0(z_3)d\xi_0(z_4) \\ &\quad + \frac{3}{4t_3^2} B_{0,0} d\xi_0(z_1)d\xi_0(z_2)d\xi_0(z_3)d\xi_0(z_4) \end{aligned} \quad (2.28)$$

and so on... Our goal, is to interpret the coefficients, like $1/24t_3$, or $-3t_5/2t_3$, or $B_{0,0}/4t_3$, in terms of intersection numbers.

3 Intersection numbers

3.1 Definitions

Let $\mathcal{M}_{g,n}$ be the moduli space of complex curves of genus g with n marked points. It is a complex orbifold (manifold quotiented by a group of symmetries), of dimension

$$\dim \mathcal{M}_{g,n} = d_{g,n} = 3g - 3 + n. \quad (3.1)$$

Let $(C, p_1, \dots, p_n) \in \mathcal{M}_{g,n}$ be a complex curve C with n marked points p_1, \dots, p_n . Let \mathcal{L}_i be the cotangent bundle at p_i , i.e. the bundle over $\mathcal{M}_{g,n}$ whose fiber is the cotangent space $T^*(p_i)$ of C at p_i . It is customary to denote its first Chern class:

$$\psi_i = c_1(\mathcal{L}_i). \quad (3.2)$$

ψ_i is (the cohomology equivalence class modulo exact forms, of) a 2-form on $\mathcal{M}_{g,n}$, therefore it makes sense to compute the "intersection number"

$$\langle \psi_1^{d_1} \dots \psi_n^{d_n} \rangle_{g,n} = \int_{[\overline{\mathcal{M}}_{g,n}]^{\text{vir}}} \psi_1^{d_1} \dots \psi_n^{d_n} \quad (3.3)$$

on the compactification $\overline{\mathcal{M}}_{g,n}$ of $\mathcal{M}_{g,n}$ (or more precisely, on a virtual cycle $[\]^{\text{vir}}$ of $\overline{\mathcal{M}}_{g,n}$, taking carefully account of the non-smooth curves at the boundary of $\mathcal{M}_{g,n}$), provided that

$$\sum_i d_i = d_{g,n} = 3g - 3 + n. \quad (3.4)$$

If this equality is not satisfied we define $\langle \psi_1^{d_1} \dots \psi_n^{d_n} \rangle_{g,n} = 0$.

More interesting characteristic classes and intersection numbers are defined as follows. Let (we follow the notations of [21], and refer the reader to it for details)

$$\pi : \overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$$

be the forgetful morphism (which forgets the last marked point), and let $\sigma_1, \dots, \sigma_n$ be the canonical sections of π , and D_1, \dots, D_n be the corresponding divisors in $\overline{\mathcal{M}}_{g,n+1}$. Let ω_π be the relative dualizing sheaf. We consider the following tautological classes on $\overline{\mathcal{M}}_{g,n}$:

- The ψ_i classes (which are 2-forms), already introduced above:

$$\psi_i = c_1(\sigma_i^*(\omega_\pi))$$

It is customary to use Witten's notation:

$$\psi_i^{d_i} = \tau_{d_i}. \quad (3.5)$$

- The Mumford κ_k classes [32, 3]:

$$\kappa_k = \pi_*(c_1(\omega_\pi(\sum_i D_i))^{k+1}).$$

κ_k is a $2k$ -form. κ_0 is the Euler class, and in $\mathcal{M}_{g,n}$, we have

$$\kappa_0 = -\chi_{g,n} = 2g - 2 + n.$$

κ_1 is known as the Weil-Petersson form since it is given by $2\pi^2\kappa_1 = \sum_i dl_i \wedge d\theta_i$ in the Fenchel-Nielsen coordinates (l_i, θ_i) in Teichmüller space [34].

In some sense, κ classes are the remnants of the ψ classes of (clusters of) forgotten points. There is the formula [3]:

$$\pi_*\psi_1^{d_1} \dots \psi_n^{d_n} \psi_{n+1}^{k+1} = \psi_1^{d_1} \dots \psi_n^{d_n} \kappa_k \quad (3.6)$$

$$\pi_*\pi_*\psi_1^{d_1} \dots \psi_n^{d_n} \psi_{n+1}^{k+1} \psi_{n+2}^{k'+1} = \psi_1^{d_1} \dots \psi_n^{d_n} (\kappa_k \kappa_{k'} + \kappa_{k+k'}) \quad (3.7)$$

and so on...

- The Hodge class $\Lambda(\alpha) = 1 + \sum_{k=1}^g (-1)^k \alpha^{-k} c_k(\mathbb{E})$ where $c_k(\mathbb{E})$ is the k^{th} Chern class of the Hodge bundle $\mathbb{E} = \pi_*(\omega_\pi)$. Mumford's formula [32, 18] says that

$$\Lambda(\alpha) = e^{\sum_{k \geq 1} \frac{B_{2k} \alpha^{1-2k}}{2k(2k-1)}} (\kappa_{2k-1} - \sum_i \psi_i^{2k-1} + \frac{1}{2} \sum_\delta \sum_j (-1)^j l_{\delta*} \psi^j \psi'^{2k-2-j}) \quad (3.8)$$

where B_k is the k^{th} Bernoulli number, δ a boundary divisor (i.e. a cycle which can be pinched so that the pinched curve is a stable nodal curve, i.e. replacing the pinched cycle by a pair of marked points, all components have a strictly negative Euler characteristics), and $l_{\delta*}$ is the natural inclusion from $\overline{\mathcal{M}}_{g,n}$ to $\overline{\mathcal{M}}_{g-1,n+2} + \sum'_{h,m} \overline{\mathcal{M}}_{h,m+1} \times \overline{\mathcal{M}}_{g-h,n-m+1}$, where $\sum'_{h,m}$ means that the sum is restricted to stable moduli spaces only. In other words $\sum_\delta l_{\delta*}$ adds a nodal point in all possible ways.

In fact, all tautological classes in $\overline{\mathcal{M}}_{g,n}$ can be expressed in terms of ψ -classes or their pull back or push forward from some $\overline{\mathcal{M}}_{h,m}$ [5]. Faber's conjecture [18] (partly proved in [31] and [21]) proposes an efficient method to compute intersection numbers of ψ , κ and Hodge classes.

3.2 Some already known cases

It is already known that :

Theorem 3.1 *If \mathcal{S} is the Airy curve $y = \sqrt{x}$, i.e. more precisely $\mathcal{S} = (\mathbb{C}, x(z) = z^2, y(z) = z, B(z, z') = dz \otimes dz' / (z - z')^2)$, one has for $2g - 2 + n > 0$*

$$W_n^{(g)}(z_1, \dots, z_n) = (-2)^{\chi_{g,n}} \sum_{d_1 + \dots + d_n = d_{g,n}} \prod_{i=1}^n \frac{(2d_i + 1)!! dz_i}{z_i^{2d_i+2}} \left\langle \prod_{i=1}^n \psi_i^{d_i} \right\rangle_{g,n}. \quad (3.9)$$

In other words the symplectic invariants of the Airy curve, generate intersection numbers of ψ classes. For the airy spectral curve, we have $F_g = 0$.

This theorem is a corollary of the following one, slightly more general:

Theorem 3.2 (proved in [31, 26, 13]) *If \mathcal{S} is the deformed Airy curve $y = \sum_k t_{k+2} x^{k/2}$, i.e. more precisely $\mathcal{S} = (\mathbb{C}, x(z) = z^2, y(z) = \sum_k t_{k+2} z^k, B(z, z') = dz \otimes dz' / (z - z')^2)$, one has for $2g - 2 + n > 0$*

$$W_n^{(g)}(z_1, \dots, z_n) = (-2)^{x_{g,n}} \sum_{d_1 + \dots + d_n \leq d_{g,n}} \prod_{i=1}^n \frac{(2d_i + 1)!! dz_i}{z_i^{2d_i+2}} \left\langle \prod_{i=1}^n \psi_i^{d_i} e^{\sum_k \tilde{t}_k \kappa_k} \right\rangle_{g,n}. \quad (3.10)$$

In particular for $n = 0$ and $g \geq 2$

$$F_g = 2^{2-2g} \left\langle e^{\sum_k \tilde{t}_k \kappa_k} \right\rangle_{g,0}, \quad (3.11)$$

where the dual times \tilde{t}_k are defined by

$$e^{-\sum_k \tilde{t}_k u^{-k}} = \sum_k (2k + 1)!! t_{2k+3} u^{-k}. \quad (3.12)$$

In other words the symplectic invariants of the deformed Airy curve, generate intersection numbers of ψ and κ classes.

proof:

This theorem can be deduced from the work of [31, 26] through Virasoro constraints. Another proof can be found in [13] by an argument similar to Kontsevich's [23], i.e. using the Strebel decomposition of the moduli space, to write those intersection numbers as expectation values of the Kontsevich matrix integral, and then computing those expectation values by integrating by parts in the matrix integral (i.e. solving loop equations). \square

Our goal is to generalize those formulae relating symplectic invariants to intersection numbers, to arbitrary spectral curves.

3.3 Main theorem

Our main theorem is

Theorem 3.3 *Let $\mathcal{S}_a = (\mathcal{C}, x, y, B)$ be a spectral curve, with only one branchpoint a . Its symplectic invariant descendents, for $2 - 2g - n < 0$, are given by the intersection*

numbers:

$$W_n^{(g)}(\mathcal{S}_a; z_1, \dots, z_n) = 2^{d_{g,n}} \sum_{d_1 + \dots + d_n \leq d_{g,n}} \prod_i d\xi_{d_i}(z_i) \left\langle e^{\frac{1}{2} \sum_{\delta} l_{\delta*} \hat{B}(\psi, \psi')} e^{\sum_k \tilde{t}_k \kappa_k} \prod_i \psi_i^{d_i} \right\rangle_{g,n} \quad (3.13)$$

In particular for $n = 0$, the symplectic invariants $F_g = W_0^{(g)}$ for $g \geq 2$ are the following intersection numbers

$$F_g(\mathcal{S}_a) = 2^{3g-3} \left\langle e^{\frac{1}{2} \sum_{\delta} l_{\delta*} \hat{B}(\psi, \psi')} e^{\sum_k \tilde{t}_k \kappa_k} \right\rangle_{g,0}. \quad (3.14)$$

In this formula:

- the times \tilde{t}_k are computed from the Laplace transform of the 1-form ydx

$$e^{-\sum_k \tilde{t}_k u^{-k}} = \frac{2 u^{3/2} e^{ux(a)}}{\sqrt{\pi}} \int_{\gamma} e^{-ux} y dx \quad (3.15)$$

where γ is a steepest descent path from the branchpoint to $x = +\infty$, i.e. $x(\gamma) - x(a) = \mathbb{R}_+$.

- the 1-forms $d\xi_d(z)$ are defined by

$$d\xi_d(z) = - \operatorname{Res}_{z' \rightarrow a} B(z, z') \frac{(2d-1)!!}{2^d (x(z') - x(a))^{d+1/2}} \quad (3.16)$$

- the kernel \hat{B}

$$\hat{B}(\psi, \psi') = \sum_{k,l} \hat{B}_{k,l} \psi^k \psi'^l \quad (3.17)$$

is defined by the double Laplace transform of the Bergman kernel:

$$\sum_{k,l} \hat{B}_{k,l} u^{-k} u'^{-l} = \frac{(uu')^{1/2} e^{(u+u')x(a)}}{2\pi} \int_{z \in \gamma} \int_{z' \in \gamma} e^{-ux(z)} e^{-u'x(z')} \left(B(z, z') - \mathring{B}(z_1, z_2) \right) \quad (3.18)$$

where the integral is regularized by subtracting the "trivial part" of the double pole

$$\mathring{B}(z_1, z_2) = \frac{dx(z_1) \otimes dx(z_2)}{4\sqrt{x(z_1) - x(a)} \sqrt{x(z_2) - x(a)}} \frac{1}{(\sqrt{x(z_1) - x(a)} - \sqrt{x(z_2) - x(a)})^2}. \quad (3.19)$$

- \sum_{δ} means the sum over all boundary divisors δ , and $l_{\delta*}$ is the "operator pinching the boundary cycle δ " to a nodal point. It adds a nodal point, i.e. two marked points, respecting stability constraints (each component must be stable, i.e. have strictly negative Euler characteristics). $l_{\delta*}$ is the natural inclusion from $\delta\mathcal{M}_{g,n}$ to $\mathcal{M}_{g-1,n+2} + \sum'_{h,m} \mathcal{M}_{h,m+1} \times \mathcal{M}_{g-h,n-m+1}$, where \sum' means that the sum is restricted

to stable moduli spaces only. ψ and ψ' are the first Chern classes of the cotangent line bundle of the nodal point.

Written in Laplace transform, Eq. (3.13) reads

$$\begin{aligned} & \prod_{i=1}^n \sqrt{\frac{\mu_i}{\pi}} e^{\mu_i x(a)} \int_{z_1 \in \gamma} \cdots \int_{z_n \in \gamma} \prod_{i=1}^n e^{-\mu_i x(z_i)} W_n^{(g)}(z_1, \dots, z_n) \\ &= 2^{d_{g,n}+n} \left\langle \prod_{i=1}^n \hat{B}(\mu_i, 1/\psi_i) e^{\frac{1}{2} \sum_{\delta} \sum_{k,l} l_{\delta*} \psi^k \psi'^l} e^{\sum_k \tilde{t}_k \kappa_k} \right\rangle_{g,n} \end{aligned} \quad (3.20)$$

where

$$\begin{aligned} \hat{B}(u, v) &= \frac{(uv)^{1/2} e^{(u+v)x(a)}}{2\pi} \int_{z \in \gamma} \int_{z' \in \gamma} e^{-ux(z)} e^{-vx(z')} B(z, z') \\ &= \frac{uv}{u+v} + \sum_{k,l} \hat{B}_{k,l} u^{-k} v^{-l} \\ &= \sum_k (-1)^k u^{k+1} v^{-k} + \sum_{k,l} \hat{B}_{k,l} u^{-k} v^{-l}. \end{aligned} \quad (3.21)$$

We shall prove this theorem below in section 6 of this article.

Before, let us see some applications.

3.4 How to use the formula

Let us show how to use the formula of theorem 3.3. First, one needs to know that ψ is a 2-form, it is assigned a degree 1, and κ_k is a $2k$ -form, which is assigned degree k . $\kappa_0 = -\chi = 2g - 2 + n$ is a number (degree 0), and can be factored out of the intersection number:

$$e^{\tilde{t}_0 \kappa_0} \rightarrow e^{(2g-2+n)\tilde{t}_0} = (2t_3)^{2-2g-n}.$$

An intersection number is non-zero only if the total degree is $d_{g,n} = 3g - 3 + n$. This means the $e^{\sum_k \tilde{t}_k \kappa_k}$ can be truncated to $k \leq 3g - 3 + n$, and the exponential can be Taylor expanded and the Taylor expansion can be truncated to order $\leq 3g - 3 + n$.

Similarly, notice that l_* diminishes $d_{g,n}$ by 1, so we may truncate the Taylor expansion of $e^{\frac{\hat{B}_{k,l}}{2} l^* \psi^k \psi'^l}$ to order $d_{g,n}$:

$$e^{\sum_{k,l} \frac{\hat{B}_{k,l}}{2} l^* \psi^k \psi'^l} \longrightarrow 1 + \sum_{j=1}^{d_{g,n}} \frac{1}{j!} \sum_{k_1, \dots, k_j, l_1, \dots, l_j} \prod_{i=1}^j \frac{\hat{B}_{k_i, l_i}}{2} l^* \psi_{n+2i-1}^{k_i} \psi_{n+2i}^{l_i}. \quad (3.22)$$

• For example for $g = 0, n = 3$, we have $d_{0,3} = 0$ and $\kappa_0 = -\chi_{0,3} = 1$, so that we may replace in $\mathcal{M}_{0,3}$

$$e^{\sum_k \tilde{t}_k \kappa_k} \longrightarrow e^{\tilde{t}_0}, \quad e^{\sum_{k,l} \frac{\hat{B}_{k,l}}{2} l^* \psi^k \psi'^l} \longrightarrow 1. \quad (3.23)$$

We thus have

$$W_3^{(0)}(z_1, z_2, z_3) = e^{\tilde{t}_0} d\xi_0(z_1) d\xi_0(z_2) d\xi_0(z_3) = \frac{1}{2t_3} d\xi_0(z_1) d\xi_0(z_2) d\xi_0(z_3) \quad (3.24)$$

which agrees with Eq. (2.26)

• For example for $g = 1, n = 1$, we have $d_{1,1} = 1$ and $\kappa_0 = -\chi_{1,1} = 1$, so that we may replace in $\mathcal{M}_{1,1}$

$$e^{\sum_k \tilde{t}_k \kappa_k} \longrightarrow e^{\tilde{t}_0} (1 + \tilde{t}_1 \kappa_1). \quad (3.25)$$

Also, since $d_{1,1} = 1$, we can replace

$$e^{\frac{\hat{B}_{k,l}}{2} l^* \psi^k \psi^l} \longrightarrow 1 + \frac{\hat{B}_{k,l}}{2} l^* \psi^k \psi^l. \quad (3.26)$$

The boundary of $\mathcal{M}_{1,1}$ is a single point, identified with $\mathcal{M}_{0,3}$. Indeed, there is only one possibility of pinching a cycle for a curve in $\mathcal{M}_{1,1}$, i.e. the torus degenerates into a sphere with 1 nodal point, i.e. a sphere with 3 marked points in $\mathcal{M}_{0,3}$. We have

$$< (l_* \psi^k \psi^l) e^{\sum_j \tilde{t}_j \kappa_j} \psi_1^{d_1} >_{1,1} = < \psi_2^k \psi_3^l e^{\sum_j \tilde{t}_j \kappa_j} \psi_1^{d_1} >_{0,3}. \quad (3.27)$$

And in $\mathcal{M}_{0,3}$, we can replace

$$e^{\sum_k \tilde{t}_k \kappa_k} \longrightarrow e^{\tilde{t}_0}. \quad (3.28)$$

Therefore, for $W_1^{(1)}$, theorem 3.3 says that:

$$\begin{aligned} \frac{1}{2} W_1^{(1)}(z) &= d\xi_1(z) \left\langle \psi e^{\tilde{t}_0 \kappa_0} \right\rangle_{1,1} + \tilde{t}_1 d\xi_0(z) \left\langle \kappa_1 e^{\tilde{t}_0 \kappa_0} \right\rangle_{1,1} \\ &\quad + \frac{\hat{B}_{0,0}}{2} d\xi_0(z) \left\langle e^{\tilde{t}_0 \kappa_0} \psi_1^0 \psi^0 \psi^0 \right\rangle_{3,0} \\ &= e^{\tilde{t}_0} \left[d\xi_1(z) \langle \psi \rangle_{1,1} + \tilde{t}_1 d\xi_0(z) \langle \kappa_1 \rangle_{1,1} + \frac{\hat{B}_{0,0}}{2} d\xi_0(z) \langle \tau_0^3 \rangle_{3,0} \right] \\ &= e^{\tilde{t}_0} \left[\frac{1}{24} d\xi_1(z) + \frac{\tilde{t}_1}{24} d\xi_0(z) + \frac{\hat{B}_{0,0}}{2} d\xi_0(z) \right] \end{aligned} \quad (3.29)$$

where we have used $< \kappa_1 >_{1,1} = 1/24$ and $< \psi >_{1,1} = 1/24$ (see appendix B). We have $\hat{B}_{0,0} = B_{0,0}/2$ and $\tilde{t}_1 = -3t_5/2t_3$, so that this expression agrees with the direct computation of Eq. (2.27).

• For example for $g = 0, n = 4$, theorem 3.3 says that:

$$\begin{aligned} \frac{1}{2} W_4^{(0)}(z_1, z_2, z_3, z_4) &= d\xi_1(z_1) d\xi_0(z_2) d\xi_0(z_3) d\xi_0(z_4) \left\langle \psi e^{\tilde{t}_0 \kappa_0} \right\rangle_{0,4} + \text{sym} \\ &\quad + \tilde{t}_1 d\xi_0(z_1) d\xi_0(z_2) d\xi_0(z_3) d\xi_0(z_4) \left\langle \kappa_1 e^{\tilde{t}_0 \kappa_0} \right\rangle_{0,4} \\ &\quad + \sum_{k,l} \frac{\hat{B}_{k,l}}{2} d\xi_0(z_1) d\xi_0(z_2) d\xi_0(z_3) d\xi_0(z_4) \left\langle \psi_1^0 \psi_2^0 \psi^k e^{\tilde{t}_0 \kappa_0} \right\rangle_{0,3} \left\langle \psi_3^0 \psi_4^0 \psi^l e^{\tilde{t}_0 \kappa_0} \right\rangle_{0,3} \end{aligned}$$

$$\begin{aligned}
& +\text{sym} \\
= & e^{2\tilde{t}_0} d\xi_1(z_1)d\xi_0(z_2)d\xi_0(z_3)d\xi_0(z_4) + \text{sym} \\
& +\tilde{t}_1 e^{2\tilde{t}_0} d\xi_0(z_1)d\xi_0(z_2)d\xi_0(z_3)d\xi_0(z_4) \\
& +\frac{6\hat{B}_{0,0}}{2} e^{2\tilde{t}_0} d\xi_0(z_1)d\xi_0(z_2)d\xi_0(z_3)d\xi_0(z_4)
\end{aligned} \tag{3.30}$$

Again, this agrees with the direct computation of Eq. (2.28).

• Similarly, for $W_2^{(1)}$, we have $d_{1,2} = 2$ and $\kappa_0 = -\chi_{1,2} = 2$, and thus

$$\begin{aligned}
\frac{1}{4}W_2^{(1)}(z_1, z_2) = & (d\xi_2(z_1)d\xi_0(z_2) + d\xi_0(z_1)d\xi_2(z_2)) \left\langle \psi^2 e^{\tilde{t}_0\kappa_0} \right\rangle_{1,2} + d\xi_1(z_1)d\xi_1(z_2) \left\langle \psi_1\psi_2 e^{\tilde{t}_0\kappa_0} \right\rangle_{1,2} \\
& +\tilde{t}_1 (d\xi_1(z_1)d\xi_0(z_2) + d\xi_0(z_1)d\xi_1(z_2)) \left\langle \psi e^{\tilde{t}_0\kappa_0} \kappa_1 \right\rangle_{1,2} \\
& +\tilde{t}_2 d\xi_0(z_1)d\xi_0(z_2) \left\langle e^{\tilde{t}_0\kappa_0} \kappa_2 \right\rangle_{1,2} + \frac{1}{2}\tilde{t}_1^2 d\xi_0(z_1)d\xi_0(z_2) \left\langle e^{\tilde{t}_0\kappa_0} \kappa_1^2 \right\rangle_{1,2} \\
& +\frac{\hat{B}_{0,0}}{2} (d\xi_1(z_1)d\xi_0(z_2) + d\xi_0(z_1)d\xi_1(z_2)) \left\langle \psi e^{\tilde{t}_0\kappa_0} \right\rangle_{0,4} \\
& +\frac{\hat{B}_{0,0}\tilde{t}_1}{2} d\xi_0(z_1)d\xi_0(z_2) \left\langle e^{\tilde{t}_0\kappa_0} \kappa_1 \right\rangle_{0,4} + \hat{B}_{1,0} d\xi_0(z_1)d\xi_0(z_2) \left\langle \psi e^{\tilde{t}_0\kappa_0} \right\rangle_{0,4} \\
& +\hat{B}_{0,0}(d\xi_1(z_1)d\xi_0(z_2) + d\xi_0(z_1)d\xi_1(z_2)) \left\langle e^{\tilde{t}_0\kappa_0} \right\rangle_{0,3} \left\langle \psi e^{\tilde{t}_0\kappa_0} \right\rangle_{1,1} \\
& +\hat{B}_{1,0}d\xi_0(z_1)d\xi_0(z_2) \left\langle e^{\tilde{t}_0\kappa_0} \right\rangle_{0,3} \left\langle \psi e^{\tilde{t}_0\kappa_0} \right\rangle_{1,1} \\
& +\hat{B}_{0,0}\tilde{t}_1 d\xi_0(z_1)d\xi_0(z_2) \left\langle e^{\tilde{t}_0\kappa_0} \right\rangle_{0,3} \left\langle e^{\tilde{t}_0\kappa_0} \kappa_1 \right\rangle_{1,1} \\
& +6\frac{\hat{B}_{0,0}^2}{8} d\xi_0(z_1)d\xi_0(z_2) \left\langle e^{\tilde{t}_0\kappa_0} \right\rangle_{0,3} \left\langle e^{\tilde{t}_0\kappa_0} \right\rangle_{0,3} \\
& +2\frac{\hat{B}_{0,0}^2}{8} d\xi_0(z_1)d\xi_0(z_2) \left\langle e^{\tilde{t}_0\kappa_0} \right\rangle_{0,3} \left\langle e^{\tilde{t}_0\kappa_0} \right\rangle_{0,3}
\end{aligned} \tag{3.31}$$

Namely:

$$\begin{aligned}
W_2^{(1)}(z_1, z_2) = & 4e^{2\tilde{t}_0} \left[\frac{1}{24} (d\xi_2(z_1)d\xi_0(z_2) + d\xi_0(z_1)d\xi_2(z_2)) + \frac{1}{24} d\xi_1(z_1)d\xi_1(z_2) \right. \\
& +\frac{\tilde{t}_1}{12} (d\xi_1(z_1)d\xi_0(z_2) + d\xi_0(z_1)d\xi_1(z_2)) \\
& +\frac{\tilde{t}_2}{24} d\xi_0(z_1)d\xi_0(z_2) + \frac{\tilde{t}_1^2}{16} d\xi_0(z_1)d\xi_0(z_2) \\
& +\frac{\hat{B}_{0,0}}{2} (d\xi_1(z_1)d\xi_0(z_2) + d\xi_0(z_1)d\xi_1(z_2)) \\
& +\frac{\hat{B}_{0,0}\tilde{t}_1}{2} d\xi_0(z_1)d\xi_0(z_2) + \hat{B}_{1,0} d\xi_0(z_1)d\xi_0(z_2) \\
& +\frac{\hat{B}_{0,0}}{24} (d\xi_1(z_1)d\xi_0(z_2) + d\xi_0(z_1)d\xi_1(z_2)) \\
& \left. +\frac{\hat{B}_{1,0}}{24} d\xi_0(z_1)d\xi_0(z_2) + \frac{\hat{B}_{0,0}\tilde{t}_1}{24} d\xi_0(z_1)d\xi_0(z_2) + \hat{B}_{0,0}^2 d\xi_0(z_1)d\xi_0(z_2) \right]
\end{aligned}$$

(3.32)

• For example for $g = 2, n = 0$, we have $d_{2,0} = 3$ and $\chi_{2,0} = -2$, and theorem 3.3 says that:

$$\begin{aligned}
\frac{1}{8}F_2 &= \left\langle e^{\sum_k \tilde{t}_k \kappa_k} \right\rangle_{2,0} \\
&+ \frac{1}{2} \sum_{i+j \leq 2} \hat{B}_{i,j} \left[\left\langle \psi_1^i \psi_2^j e^{\sum_k \tilde{t}_k \kappa_k} \right\rangle_{1,2} + \left\langle \psi_1^i e^{\sum_k \tilde{t}_k \kappa_k} \right\rangle_{1,1} \left\langle \psi_2^j e^{\sum_k \tilde{t}_k \kappa_k} \right\rangle_{1,1} \right] \\
&+ \frac{1}{8} \sum_{i+j+m+n \leq 1} \hat{B}_{i,j} \hat{B}_{m,n} \left[\left\langle \psi_1^i \psi_2^j \psi_3^m \psi_4^n e^{\sum_k \tilde{t}_k \kappa_k} \right\rangle_{0,4} \right. \\
&+ 2 \left\langle \psi_1^i \psi_2^j \psi_3^m e^{\sum_k \tilde{t}_k \kappa_k} \right\rangle_{0,3} \left\langle \psi_4^n e^{\sum_k \tilde{t}_k \kappa_k} \right\rangle_{1,1} \\
&+ 2 \left\langle \psi_1^i \psi_3^m \psi_4^n e^{\sum_k \tilde{t}_k \kappa_k} \right\rangle_{0,3} \left\langle \psi_2^j e^{\sum_k \tilde{t}_k \kappa_k} \right\rangle_{1,1} \left. \right] \\
&+ \frac{1}{48} \sum_{i+j+m+n+p+q \leq 0} \hat{B}_{i,j} \hat{B}_{m,n} \hat{B}_{p,q} \left[\right. \\
&2 \left\langle \psi_1^i \psi_2^j \psi_5^p e^{\sum_k \tilde{t}_k \kappa_k} \right\rangle_{0,3} \left\langle \psi_3^m \psi_4^n \psi_6^q e^{\sum_k \tilde{t}_k \kappa_k} \right\rangle_{0,3} \\
&+ 2 \left\langle \psi_1^i \psi_3^m \psi_5^p e^{\sum_k \tilde{t}_k \kappa_k} \right\rangle_{0,3} \left\langle \psi_2^j \psi_4^n \psi_6^q e^{\sum_k \tilde{t}_k \kappa_k} \right\rangle_{0,3} \\
&+ 2 \left\langle \psi_1^i \psi_4^n \psi_5^p e^{\sum_k \tilde{t}_k \kappa_k} \right\rangle_{0,3} \left\langle \psi_2^j \psi_3^m \psi_6^q e^{\sum_k \tilde{t}_k \kappa_k} \right\rangle_{0,3} \\
&+ 2 \left\langle \psi_1^i \psi_2^j \psi_3^m e^{\sum_k \tilde{t}_k \kappa_k} \right\rangle_{0,3} \left\langle \psi_4^n \psi_5^p \psi_6^q e^{\sum_k \tilde{t}_k \kappa_k} \right\rangle_{0,3} \\
&+ 2 \left\langle \psi_1^i \psi_3^m \psi_4^n e^{\sum_k \tilde{t}_k \kappa_k} \right\rangle_{0,3} \left\langle \psi_2^j \psi_5^p \psi_6^q e^{\sum_k \tilde{t}_k \kappa_k} \right\rangle_{0,3} \left. \right] \\
\end{aligned} \tag{3.33}$$

Namely, that gives (see appendix B)

$$\begin{aligned}
F_2 &= 8e^{2\tilde{t}_0} \left\{ \frac{\tilde{t}_3}{3^2 2^7} + \frac{\tilde{t}_2 \tilde{t}_1}{15 2^4} + \frac{43 \tilde{t}_1^3}{5 3^3 2^7} + \frac{\hat{B}_{0,0}}{2} \left(\frac{\tilde{t}_2}{24} + \frac{\tilde{t}_1^2}{16} \right) + \frac{\hat{B}_{1,0} \tilde{t}_1}{12} + \frac{\hat{B}_{1,1}}{48} + \frac{\hat{B}_{2,0}}{48} \right. \\
&+ \frac{\hat{B}_{0,0} \tilde{t}_1^2}{2 * 24^2} + \frac{\hat{B}_{1,0} \tilde{t}_1}{24^2} + \frac{\hat{B}_{1,1}}{2 * 24^2} + \frac{\hat{B}_{2,0} \tilde{t}_1}{8} \left(1 + \frac{4}{24} \right) + \frac{\hat{B}_{0,0} \hat{B}_{1,0}}{8} \left(4 + \frac{4}{24} \right) \\
&\left. + \frac{10 \hat{B}_{0,0}^3}{48} \right\}. \tag{3.34}
\end{aligned}$$

One can check that this agrees with the direct computation of symplectic invariants, computing the residues in def. 2.4.

4 The topological vertex

Specializing theorem 3.3 to the topological vertex's [1, 20, 27] spectral curve we get:

Theorem 4.1 (Topological vertex and BKMP) *For any framing f , choose the framed topological vertex spectral curve $\mathcal{S}_{\text{vertex}} = (\mathbb{C}^* \setminus]-\infty, 0] \cup [1, \infty[, x(z) = -f \ln z - \ln(1-z), y(z) = -\ln z, B(z, z') = dz dz' / (z - z')^2)$, i.e.:*

$$e^{-x} = e^{-fy} (1 - e^{-y}).$$

Then we have

$$W_n^{(g)}(\mathcal{S}_{\text{vertex}}; z_1, \dots, z_n) = 2^{d_{g,n}} \sum_{d_1 + \dots + d_n \leq 3g-3+n} \prod_i d\hat{\xi}_{d_i}(z_i) \left\langle \Lambda(1)\Lambda(f)\Lambda(-1-f) \prod_i \psi_i^{d_i} \right\rangle_{g,n} \quad (4.1)$$

where $\Lambda(\alpha) = 1 + \sum_{k=1}^g (-1)^k \alpha^{-k} c_k(\mathbb{E})$ is the Hodge class ($c_k(\mathbb{E})$ is the k^{th} Chern class of the Hodge bundle $\mathbb{E} = \pi_*(\omega_\pi)$), and

$$\xi_d(z) = \frac{f+1}{f} \sum_{\mu=0}^{\infty} e^{-\mu x(z)} \frac{(-\mu)^d (\mu(1+1/f))!}{\mu! (\mu/f)!} \quad (4.2)$$

In other words we recognize Mariño–Vafa formula [29] for the topological vertex

$$W_n^{(g)}(\mathcal{S}_{\text{vertex}}; z_1, \dots, z_n) = \sum_{\mu_1, \dots, \mu_n} \prod_i \frac{(\mu_i(f+1))!}{\mu_i! (f\mu_i)!} e^{-\mu_i x(z_i)} \mu_i dx(z_i) \left\langle \frac{\Lambda(1)\Lambda(f)\Lambda(-1-f)}{\prod_i 1 + \mu_i \psi_i} \right\rangle_{g,n}. \quad (4.3)$$

This gives a new proof of the BKMP conjecture [7] for the vertex (already proved in [9, 36]), i.e. that the symplectic invariants of the framed vertex spectral curve $\mathcal{S}_{\text{vertex}}$ (which is the mirror curve of \mathbb{C}^3), are the Gromov-Witten invariants of the framed vertex.

proof:

We prove this theorem in section 7 below. We just mention that this theorem is already known from [9, 36]. We just propose a new proof using only the topological recursion. \square

5 Several branchpoints

Theorem 3.3 Immediately generalizes to several branchpoints.

Theorem 5.1 *Let $\mathcal{S} = (\mathcal{C}, x, y, B)$ be a spectral curve with branchpoints a_1, \dots, a_ℓ . Its symplectic invariant descendents of \mathcal{S} are given by*

$$W_n^{(g)}(\mathcal{S}; z_1, \dots, z_n) = 2^{d_{g,n}} \sum_{d_1 + \dots + d_n \leq d_{g,n}} \sum_m \sum_{\deg. \mathcal{M}_{g,n} \rightarrow \cup_{j=1}^m \mathcal{M}_{g_j, n_j + k_j}} \sum_{1 \leq a_j \leq \ell, j=1, \dots, m}$$

$$\sum_{I_1 \uplus \dots \uplus I_m = J, \#I_j = n_j} \sum_{d_{j,i}, i=1, \dots, k_j} \frac{1}{\#\text{Aut}} \prod_{j < j'}^{k_j} \prod_{i=1}^{k_j} \prod_{i'=1}^{k_{j'}} \hat{B}_{a_j, d_{j,i}; a_{j'}, d_{j',i'}} \prod_{j=1}^m \prod_{z_i \in I_j} d\xi_{a_j, d_i}(z_i) \left\langle \Lambda_{a_j} \prod_{z_i \in I_j} \tau_{d_i} \prod_{i=1}^{k_j} \tau_{d_{j,i}} \right\rangle_{g_j, n_j + k_j} \quad (5.1)$$

where $J = \{z_1, \dots, z_n\}$ and we sum over all stable degeneracies of $\mathcal{M}_{g,n}$ made of m stable connected components, the j^{th} component having genus g_j , having n_j marked points, and k_j nodal points.

We have defined

- The forms $d\xi_{a,d}(z)$ for each branchpoint a

$$d\xi_{a,d}(z) = - \operatorname{Res}_{z' \rightarrow a} B(z, z') \frac{(2d-1)!!}{2^d (x(z') - x(a))^{d+1/2}} \quad (5.2)$$

- The double Laplace transforms of the Bergman kernel

$$\begin{aligned} & \sum_{k, k'} \hat{B}_{a, k; a', k'} u^{-k} v^{-k'} \\ &= (1 - \delta_{a, a'}) \frac{\sqrt{uv}}{2\pi} \int_{z \in \gamma_a} e^{-u(x(z) - x(a))} \int_{z' \in \gamma_{a'}} e^{-v(x(z') - x(a'))} B(z, z') \end{aligned} \quad (5.3)$$

- The tautological class Λ_a associated to the branchpoint a :

$$\Lambda_a = e^{\sum_k \tilde{t}_{a,k} \kappa_k} e^{\frac{1}{2} \sum_{\delta} \sum_{k,l} \tilde{B}_{a,k;a,l} l_{\delta} \psi^k \psi^l} \quad (5.4)$$

where \tilde{t}_k are the dual times

$$e^{-\sum_k \tilde{t}_{a,k} u^{-k}} = \frac{2\sqrt{u} e^{ux(a)}}{\sqrt{\pi}} \int_{\gamma_a} e^{-ux(z)} dy(z) \quad (5.5)$$

where the steepest descent contour γ_a for a branchpoint a , is a connected arc on \mathcal{C} , going through a , and such that $x(\gamma_a) - x(a) = \mathbb{R}_+$. And the $\hat{B}_{a,k,k'}$ are given by

$$\begin{aligned} & \sum_{k, k'} \hat{B}_{a,k,k'} u^{-k} v^{-k'} \\ &= \frac{\sqrt{uv}}{\pi} \int_{z \in \gamma_a} e^{-u(x(z) - x(a))} \int_{z' \in \gamma_a} e^{-v(x(z') - x(a))} (B(z, z') - \mathring{B}_a(z, z')). \end{aligned} \quad (5.6)$$

proof:

This theorem is the immediate generalization of theorem 3.3, using the methods of [33, 24], or an immediate generalization of lemma 6.1 proved in appendix D. \square

6 Proof of the main theorem

Let us prove theorem 3.3.

6.1 Kontsevich's curve Symplectic invariants

Consider a spectral curve $\mathcal{S}_a = (\mathcal{C}, x, y, B)$ where \mathcal{C} contains only one branch-point located at a . Locally we write the Taylor expansion near $x = x(a)$ as:

$$y \sim_a \sum_k t_{k+2} (x - x(a))^{\frac{k}{2}}. \quad (6.1)$$

The Bergman kernel $B(z_1, z_2)$ is used to define symplectic invariants. The Bergman kernel is a symmetric 2-form on $\mathcal{S}_a \times \mathcal{S}_a$ with a double pole on the diagonal:

$$B(z_1, z_2) \sim \frac{dz_1 \otimes dz_2}{(z_1 - z_2)^2} + \text{regular} \quad (6.2)$$

where z may be any local parameter on \mathcal{S}_a , in particular, if we choose $z = \zeta = \sqrt{x - x(a)}$ near the branchpoint a , and denote

$$\begin{aligned} \mathring{B}(z_1, z_2) &= \frac{1}{4 \sqrt{(x_1 - x(a))(x_2 - x(a))}} \frac{dx_1 \otimes dx_2}{(\sqrt{x_1 - x(a)} - \sqrt{x_2 - x(a)})^2} \\ &= \frac{d\zeta(z_1) \otimes d\zeta(z_2)}{(\zeta(z_1) - \zeta(z_2))^2}, \end{aligned} \quad (6.3)$$

we have that

$$B(z_1, z_2) - \mathring{B}(z_1, z_2) = \text{analytical near } z_1 \rightarrow a, z_2 \rightarrow a. \quad (6.4)$$

Let us now consider the same spectral curve with the Bergman kernel \mathring{B} .

$$\mathring{\mathcal{S}}_a = (\mathcal{C}, x, y, \mathring{B}). \quad (6.5)$$

$\mathring{\mathcal{S}}_a$ is the Kontsevich's spectral curve, and thus according to theorem 3.2 (proved for instance in [13]), the symplectic invariants $\mathring{W}_n^{(g)} = W_n^{(g)}(\mathring{\mathcal{S}}_a)$, are (with $\zeta_i = \sqrt{x(z_i) - x(a)}$):

$$\mathring{W}_n^{(g)}(\zeta_1, \dots, \zeta_n) = (-1)^n 2^{d_{g,n}} \sum_{d_1 + \dots + d_n \leq d_{g,n}} \prod_{i=1}^n \frac{(2d_i + 1)!!}{2^{d_i} \zeta_i^{2d_i+2}} d\zeta_i \left\langle e^{\sum_k \tilde{t}_k \kappa_k} \prod_{i=1}^n \psi_i^{d_i} \right\rangle_{g,n} \quad (6.6)$$

where the times \tilde{t}_k are the Schur transforms of the t_k 's, defined through their generating function:

$$e^{-\sum_{k=0}^{\infty} \tilde{t}_k u^{-k}} = 2 \sum_{k=0}^{\infty} \frac{(2k+1)!!}{2^k} t_{2k+3} u^{-k}. \quad (6.7)$$

6.2 Laplace transform and the spectral curve class

We thus see that we are led to associate to any spectral curve \mathcal{S} with one branchpoint a , the following tautological class

$$e^{\sum_{k=0}^{\infty} \tilde{t}_k \kappa_k} = e^{\tilde{t}_0 \kappa_0} \left(1 + \tilde{t}_1 \kappa_1 + \left(\frac{\tilde{t}_1^2}{2} \kappa_1^2 + \tilde{t}_2 \kappa_2 \right) + \left(\tilde{t}_3 \kappa_3 + \tilde{t}_1 \tilde{t}_2 \kappa_1 \kappa_2 + \frac{\tilde{t}_1^3}{6} \kappa_1^3 \right) + \dots \right) \quad (6.8)$$

where the times \tilde{t}_k are determined by the generating function

$$G(u) = e^{-g(u)} = e^{-\sum_{k=0}^{\infty} \tilde{t}_k u^{-k}} = 2 \sum_{k=0}^{\infty} (2k+1)!! t_{2k+3} 2^{-k} u^{-k}. \quad (6.9)$$

Notice that we have:

$$\begin{aligned} \int_{x=x(a)}^{\infty} e^{-ux} (y - \bar{y}) dx &= 2 e^{-ux(a)} \sum_{k=0}^{\infty} \int_{x(a)}^{\infty} t_{2k+3} (x - x(a))^{k+1/2} e^{-u(x-x(a))} dx \\ &= 2 e^{-ux(a)} \sum_{k=0}^{\infty} t_{2k+3} \int_0^{\infty} \zeta^{2k+1} e^{-u\zeta^2} 2\zeta d\zeta \\ &= 2 e^{-ux(a)} \sum_{k=0}^{\infty} t_{2k+3} \int_{-\infty}^{\infty} \zeta^{2k+2} e^{-u\zeta^2} d\zeta \\ &= 2 e^{-ux(a)} \sum_{k=0}^{\infty} t_{2k+3} (2k+1)!! 2^{-k-1} u^{-k-1} \sqrt{\frac{\pi}{u}} \\ &= \frac{1}{2} \sqrt{\pi} u^{-3/2} e^{-ux(a)} e^{-g(u)}. \end{aligned} \quad (6.10)$$

In other words, the $G(u) = e^{-g(u)}$ function is related to the Laplace transform of $y dx$ along a contour passing through the branchpoint:

$$G(u) = e^{-g(u)} = 2 \frac{u^{3/2} e^{ux(a)}}{\sqrt{\pi}} \int_{\gamma_a} e^{-ux} y dx \quad (6.11)$$

Here, γ_a is a contour on the spectral curve, passing through the branchpoint a , and whose x projection is:

$$x(\gamma_a) = [x(a), +\infty[\quad (6.12)$$

Let us also integrate by parts:

$$G(u) = e^{-g(u)} = -2 \frac{u^{3/2} e^{ux(a)}}{\sqrt{\pi}} \frac{1}{u} \int_{\gamma_a} y d(e^{-ux}) \quad (6.13)$$

i.e.

$$G(u) = e^{-g(u)} = 2 \frac{u^{1/2} e^{ux(a)}}{\sqrt{\pi}} \int_{\gamma_a} e^{-ux} dy. \quad (6.14)$$

6.3 Introducing the Bergman kernel

So far, we have computed $\overset{\circ}{W}_n^{(g)}$ with the Bergman kernel $\overset{\circ}{B}$, and not with the proper Bergman kernel B of the spectral curve \mathcal{S} . we now need to reintroduce the correct Bergman kernel.

First, using the local variable $\zeta(z) = \sqrt{x(z) - x(a)}$, compute the Taylor expansion of $B(z_1, z_2)$ near a :

$$B(z_1, z_2) - \overset{\circ}{B}(z_1, z_2) = \sum_{k,l} B_{k,l} \zeta(z_1)^k \zeta(z_2)^l d\zeta(z_1) d\zeta(z_2). \quad (6.15)$$

6.4 The basis $d\xi_d$

We introduce the differential forms:

$$d\xi_d(z) = -\frac{(2d-1)!!}{2^d} \operatorname{Res}_{z' \rightarrow a} B(z, z') \frac{1}{\zeta(z')^{2d+1}}. \quad (6.16)$$

They are defined globally on the Riemann surface \mathcal{C} , and they have poles only at the branch point. We also introduce the even forms

$$d\tilde{\xi}_d(z) = -\operatorname{Res}_{z' \rightarrow a} B(z, z') \frac{1}{\zeta(z')^{2d}}. \quad (6.17)$$

In the vicinity of the branchpoint we have the Laurent series expansion, using Eq. (6.15):

$$d\xi_d(z) = -\frac{(2d+1)!!}{2^d} \frac{d\zeta(z)}{\zeta(z)^{2d+2}} - \frac{(2d-1)!!}{2^d} \sum_k B_{2d,k} \zeta(z)^k d\zeta(z) \quad (6.18)$$

i.e.

$$\xi_d(z) = \frac{(2d-1)!!}{2^d} \left(\frac{1}{\zeta(z)^{2d+1}} - \sum_k B_{2d,k} \frac{\zeta(z)^{k+1}}{k+1} \right). \quad (6.19)$$

And similarly

$$d\tilde{\xi}_d(z) = -2d \frac{d\zeta(z)}{\zeta(z)^{2d+1}} - \sum_k B_{2d-1,k} \zeta(z)^k d\zeta(z). \quad (6.20)$$

They are such that when z_1 is in the vicinity of the branchpoint (but not necessarily z_2):

$$\begin{aligned} B(z_1, z_2) &= -\sum_{d=0}^{\infty} \zeta(z_1)^{2d} d\zeta(z_1) \frac{2^d}{(2d-1)!!} d\xi_d(z_2) \\ &\quad - \sum_{d=0}^{\infty} \zeta(z_1)^{2d-1} d\zeta(z_1) d\tilde{\xi}_d(z_2). \end{aligned} \quad (6.21)$$

Since all $W_n^{(g)}$'s are computed by taking residues at the branchpoint, we always need to replace $B(z_1, z_2)$ by its Taylor expansion, and therefore, $W_n^{(g)}$ is a linear combination of the $d\xi_{d_i}(z_i)$ and $d\tilde{\xi}_{d_i}(z_i)$. Moreover, it is a known property (see [16]) of $W_n^{(g)}$ that

$$W_n^{(g)}(z_1, z_2, \dots, z_n) + W_n^{(g)}(\bar{z}_1, z_2, \dots, z_n) \quad (6.22)$$

is analytical when z_1 is at the branchpoint, i.e. there can be only odd degree poles in $\zeta(z_1)$, and thus the even $d\tilde{\xi}_{d_i}(z_i)$ don't appear in $W_n^{(g)}$.

Therefore $W_n^{(g)}$ can be decomposed uniquely on that basis, as:

$$W_n^{(g)}(z_1, \dots, z_n) = 2^{d_{g,n}} \sum_{d_1 + \dots + d_n \leq 3g-3+n} A_n^{(g)}(d_1, \dots, d_n) \prod_{i=1}^n d\xi_{d_i}(z_i). \quad (6.23)$$

We have

$$\text{Res}_{z \rightarrow a} \zeta(z)^{2k+1} d\xi_a(z) = -(2d+1)!! 2^{-d} \delta_{k,d}, \quad (6.24)$$

so that

$$2^{d_{g,n}} A_n^{(g)}(d_1, \dots, d_n) = (-1)^n \text{Res}_{z_i \rightarrow a} W_n^{(g)}(z_1, \dots, z_n) \prod_{i=1}^n \frac{2^{d_i} \zeta(z_i)^{2d_i+1}}{(2d_i+1)!!}. \quad (6.25)$$

In particular for Kontsevich integral, i.e. with $B = \mathring{B}$ we have

$$\mathring{A}_n^{(g)}(d_1, \dots, d_n) = \left\langle e^{\sum_k \tilde{t}_k \kappa_k} \prod_{i=1}^n \psi_i^{d_i} \right\rangle_{g,n}. \quad (6.26)$$

6.5 Lemma

Lemma 6.1 *Let $J = \{d_1, \dots, d_n\}$, and let $2 - 2g - n < 0$, we have*

$$\begin{aligned} 2 \frac{\partial W_n^{(g)}(J)}{\partial B_{k,l}} &= \frac{1}{(k+1)(l+1)} \text{Res}_{z \rightarrow \infty} \text{Res}_{z' \rightarrow \infty} z^{k+1} z'^{l+1} \left[W_{n+2}^{(g-1)}(z, z', J) \right. \\ &\quad + \sum_h \sum_{I \subset J}^{\text{stable}} W_{1+\#I}^{(h)}(z, I) W_{1+n-\#I}^{(g-h)}(z', J \setminus I) \\ &\quad \left. + 2 \sum_{z_i \in J} W_2^{(0)}(z, z_i) W_n^{(g)}(z', J \setminus \{z_i\}) \right] \end{aligned} \quad (6.27)$$

which implies that

$$\begin{aligned} \frac{\partial A_n^{(g)}(J)}{\partial B_{2k,2l}} &= 2^{-k-l-1} (2k-1)!! (2l-1)!! \left[A_{n+2}^{(g-1)}(k, l, J) \right. \\ &\quad \left. + \sum_h \sum_{I \subset J}^{\text{stable}} A_{1+\#I}^{(h)}(k, I) A_{1+n-\#I}^{(g-h)}(l, J \setminus I) \right]. \end{aligned} \quad (6.28)$$

Moreover the derivatives of $A_n^{(g)}$ with respect to $B_{k,l}$ where k or l is odd vanish.

We shall denote

$$\hat{B}_{k,l} = (2k-1)!! (2l-1)!! 2^{-k-l-1} B_{2k,2l}. \quad (6.29)$$

proof:

We present a self contained proof of the first equation of this lemma in appendix D below, but we mention that this lemma is a straightforward application of the formalism of Kostov and Orantin [24, 33], based on earlier work of Kostov, and related to the Givental formalism.

It can also be seen as a very simple generalization of the "holomorphic anomaly equations as in [14, 16]. Let us recall that in [16], modular transformations of the spectral curve amount to change the Bergman kernel as:

$$B(z_1, z_2) \rightarrow B(z_1, z_2) + \sum_{k,l} c_{k,l} du_k(z_1) du_l(z_2) \quad (6.30)$$

where du_k , $k = 1, \dots$, genus, are the holomorphic forms on the spectral curve, satisfying:

$$du_k(z) = \frac{1}{2i\pi} \oint_{z' \in \mathcal{B}_k} B(z, z'), \quad (6.31)$$

and it was found in [16] that

$$\begin{aligned} 2 \frac{\partial W_n^{(g)}(J)}{\partial c_{k,l}} &= \frac{1}{(2i\pi)^2} \oint_{z \in \mathcal{B}_k} \oint_{z' \in \mathcal{B}_l} \left[W_{n+2}^{(g-1)}(z, z', J) \right. \\ &\quad \left. + \sum_h \sum_{I \subset J}^{\text{stable}} W_{1+\#I}^{(h)}(z, I) W_{1+n-\#I}^{(g-h)}(z', J \setminus I) \right]. \end{aligned} \quad (6.32)$$

It can be seen that the derivation of [16] doesn't rely on the fact that du_k are holomorphic, it works for du_k meromorphic, and thus the present Lemma is an analogous of this when $du_k(z)$ are of the form $du_k = \zeta^k d\zeta$.

Therefore, similarly to Eq. (6.30), we write (doing as if the sum over k and l were finite), and using the local parameter $z = \zeta = \sqrt{x(z) - x(a)}$:

$$\begin{aligned} B(z_1, z_2) - \overset{\circ}{B}(z_1, z_2) &= \sum_{k,l} B_{k,l} z_1^k z_2^l dz_1 dz_2 \\ &= \sum_{k,l} B_{k,l} \operatorname{Res}_{z \rightarrow \infty} \operatorname{Res}_{z' \rightarrow \infty} \frac{z^{k+1}}{k+1} B(z, z_1) \frac{z'^{l+1}}{l+1} B(z', z_2) \end{aligned} \quad (6.33)$$

and, similarly to Eq. (6.32), we get

$$2 \frac{\partial W_n^{(g)}(J)}{\partial B_{k,l}} = \frac{1}{(k+1)(l+1)} \operatorname{Res}_{z \rightarrow \infty} \operatorname{Res}_{z' \rightarrow \infty} z^{k+1} z'^{l+1} \left[W_{n+2}^{(g-1)}(z, z', J) + \sum_h \sum_{I \subset J} W_{1+\#I}^{(h)}(z, I) W_{1+n-\#I}^{(g-h)}(z', J \setminus I) \right] \quad (6.34)$$

where \sum' excludes all cases where one of the factors is $W_1^{(0)}$.

This was just a sketch of the proof, a full self contained proof of this equation is presented in appendix D.

Now, let us prove the second part of the Lemma.

Notice that the sum in the right hand side of Eq. (6.34) includes cases where one of the factors is $W_2^{(0)}$, i.e. we write

$$2 \frac{\partial W_n^{(g)}(J)}{\partial B_{k,l}} = \frac{1}{(k+1)(l+1)} \operatorname{Res}_{z \rightarrow \infty} \operatorname{Res}_{z' \rightarrow \infty} z^{k+1} z'^{l+1} \left[W_{n+2}^{(g-1)}(z, z', J) + \sum_h \sum_{I \subset J}^{\text{stable}} W_{1+\#I}^{(h)}(z, I) W_{1+n-\#I}^{(g-h)}(z', J \setminus I) + 2 \sum_{z_i \in J} W_2^{(0)}(z, z_i) W_n^{(g)}(z', J \setminus \{z_i\}) \right] \quad (6.35)$$

where now \sum^{stable} means both factors must be stable, i.e. we exclude all terms where one factor is either $W_1^{(0)}$ or $W_2^{(0)}$.

Since $W_2^{(0)}(z, z_i) = B(z, z_i)$, the residues of the last term give

$$2 \frac{\partial W_n^{(g)}(J)}{\partial B_{k,l}} = \frac{1}{(k+1)(l+1)} \operatorname{Res}_{z \rightarrow \infty} \operatorname{Res}_{z' \rightarrow \infty} z^{k+1} z'^{l+1} \left[W_{n+2}^{(g-1)}(z, z', J) + \sum_h \sum_{I \subset J}^{\text{stable}} W_{1+\#I}^{(h)}(z, I) W_{1+n-\#I}^{(g-h)}(z', J \setminus I) \right] + \frac{2}{(l+1)} \sum_{z_i \in J} z_i^k dz_i \operatorname{Res}_{z' \rightarrow \infty} z'^{l+1} W_n^{(g)}(z', J \setminus \{z_i\}) \quad (6.36)$$

If we write that

$$W_n^{(g)}(z_1, \dots, z_n) = 2^{d_{g,n}} \sum_{d_1 + \dots + d_n \leq 3g-3+n} A_n^{(g)}(d_1, \dots, d_n) \prod_{i=1}^n d\xi_{d_i}(z_i) \quad (6.37)$$

and using that

$$\operatorname{Res}_{z' \rightarrow \infty} z'^{2k+1} d\xi_d(z') = - \operatorname{Res}_{z' \rightarrow 0} z'^{2k+1} d\xi_d(z') = (2d+1)!! 2^{-d} \delta_{k,d}, \quad (6.38)$$

and

$$\text{Res}_{z' \rightarrow \infty} z'^{2k} d\xi_d(z') = 0, \quad (6.39)$$

we find

$$\begin{aligned} & 4 \sum_{d_1, \dots, d_n} \left[\frac{\partial A_n^{(g)}(d_1, \dots, d_n)}{\partial B_{k,l}} \prod_i d\xi_{d_i}(z_i) + \sum_i A_n^{(g)}(d_1, \dots, d_n) \frac{\partial d\xi_{d_i}(z_i)}{\partial B_{k,l}} \prod_{j \neq i} d\xi_{d_j}(z_j) \right] \\ = & \sum_{d, d'} \sum_{d_1, \dots, d_n} \delta_{k,2d} \delta_{l,2d'} (2d-1)!! (2d'-1)!! 2^{-d-d'} \left[A_{n+2}^{(g-1)}(d, d', d_1, \dots, d_n) \right. \\ & + \sum_h \sum_{I \subset \{d_1, \dots, d_n\}}^{\text{stable}} A_{1+\#I}^{(h)}(d, I) A_{1+n-\#I}^{(g-h)}(d', \{d_1, \dots, d_n\} \setminus I) \left. \right] \prod_{i=1}^n d\xi_{d_i}(z_i) \\ & + 4 \sum_{i=1}^n z_i^k dz_i \sum_{d'} \sum_{d_1, \dots, d_n} \delta_{l,2d'} (2d'-1)!! 2^{-d'} A_n^{(g)}(d', \{d_1, \dots, d_n\} \setminus \{d_i\}) \prod_{j \neq i} d\xi_{d_j}(z_j) \end{aligned} \quad (6.40)$$

The last line exactly simplifies with the second term in the first line, and thus we get:

$$\begin{aligned} & 2 \frac{\partial A_n^{(g)}(d_1, \dots, d_n)}{\partial B_{k,l}} \prod_i d\xi_{d_i}(z_i) \\ = & \sum_{d, d'} \delta_{k,2d} \delta_{l,2d'} (2d-1)!! (2d'-1)!! 2^{-d-d'-1} \left[A_{n+2}^{(g-1)}(d, d', d_1, \dots, d_n) \right. \\ & + \sum_h \sum_{I \subset \{d_1, \dots, d_n\}}^{\text{stable}} A_{1+\#I}^{(h)}(d, I) A_{1+n-\#I}^{(g-h)}(d', \{d_1, \dots, d_n\} \setminus I) \left. \right] \end{aligned} \quad (6.41)$$

which is the Lemma.

□

At $B = \overset{\circ}{B}$, i.e. at $\hat{B}_{k,l} = 0$, we have

$$\overset{\circ}{A}_n^{(g)}(d_1, \dots, d_n) = \langle \psi_1^{d_1} \dots \psi_n^{d_n} e^{\sum_k \tilde{t}_k \kappa_k} \rangle_{g,n} \quad (6.42)$$

and thus

$$\begin{aligned} & 2 \frac{\partial}{\partial \hat{B}_{k,l}} A_n^{(g)}(J) \Big|_{\hat{B}_{k,l}=0} = \langle \psi_{n+1}^k \psi_{n+2}^l \prod_{i \in J} \psi_i^{d_i} e^{\sum_k \tilde{t}_k \kappa_k} \rangle_{g-1, n+2} \\ & + \sum_h \sum_{I \subset J}^{\text{stable}} \langle \psi_{n+1}^k \prod_{i \in I} \psi_i^{d_i} e^{\sum_k \tilde{t}_k \kappa_k} \rangle_{h, 1+\#I} \langle \psi_{n+2}^l \prod_{i \notin I} \psi_i^{d_i} e^{\sum_k \tilde{t}_k \kappa_k} \rangle_{g-h, 1+n-\#I} \Big]. \\ = & \left\langle \sum_{\delta} l_{\delta*} \psi_{n+1}^k \psi_{n+2}^l \prod_{i \in J} \psi_i^{d_i} e^{\sum_k \tilde{t}_k \kappa_k} \right\rangle_{g,n}. \end{aligned} \quad (6.43)$$

Similarly, computing the m^{th} derivative at $B = \overset{\circ}{B}$ we get:

$$2^m \frac{\partial^m}{\partial \hat{B}_{k_1, l_1} \dots \partial \hat{B}_{k_m, l_m}} A_n^{(g)}(J) \Big|_{\hat{B}_{k, l}=0} = \left\langle \prod_{r=1}^m \left(\sum_{\delta} l_{\delta*} \psi_{n+2r-1}^{k_r} \psi_{n+2r}^{l_r} \right) \prod_{i \in J} \psi_i^{d_i} e^{\sum_k \tilde{t}_k \kappa_k} \right\rangle_{g,n} \quad (6.44)$$

And thus, by writing the Taylor expansion we get

$$A_n^{(g)}(d_1, \dots, d_n) = \langle \psi_1^{d_1} \dots \psi_n^{d_n} e^{\sum_k \tilde{t}_k \kappa_k} e^{\frac{1}{2} l_* \hat{B}(\psi, \psi')} \rangle_{g,n}, \quad (6.45)$$

where

$$\hat{B}(\psi, \psi') = \sum_{k,l} \hat{B}_{k,l} \psi^k \psi'^l, \quad (6.46)$$

and $l_* = \sum_{\delta} l_{\delta*}$ is the projection to all boundary divisors δ .

This ends the proof of theorem 3.3.

6.6 Change of basis

It is sometimes good idea to change the basis $d\xi_d$ to another basis.

$$d\xi_d = \sum_{d' \leq d} C_{d, d-d'} d\xi_{d'}. \quad (6.47)$$

That gives

$$\begin{aligned} & 2^{-d_{g,n}} W_n^{(g)}(z_1, \dots, z_n) \\ &= \sum_{d_i} \prod_i d\xi_{d_i}(z_i) \left\langle \prod_i \psi_i^{d_i} e^{\frac{1}{2} \sum_{k,l} \hat{B}_{k,l} l_* \psi^k \psi'^l} e^{\sum_k \tilde{t}_k \kappa_k} \right\rangle_{g,n} \\ &= \sum_{d_i, d'_i} \prod_i C_{d_i, d_i - d'_i} d\hat{\xi}_{d'_i}(z_i) \left\langle \prod_i \psi_i^{d_i} e^{\frac{1}{2} \sum_{k,l} \hat{B}_{k,l} l_* \psi^k \psi'^l} e^{\sum_k \tilde{t}_k \kappa_k} \right\rangle_{g,n} \\ &= \sum_{d'_i} \prod_i d\hat{\xi}_{d'_i}(z_i) \left\langle \prod_i \psi_i^{d'_i} \left(\sum_{d_i} C_{d_i + d'_i, d_i} \psi_i^{d_i} \right) e^{\frac{1}{2} \sum_{k,l} \hat{B}_{k,l} l_* \psi^k \psi'^l} e^{\sum_k \tilde{t}_k \kappa_k} \right\rangle_{g,n} \end{aligned} \quad (6.48)$$

and therefore it is interesting to introduce the functions

$$f_d(u) = u^d \sum_k C_{d+k} u^k \quad (6.49)$$

that gives

$$W_n^{(g)}(z_1, \dots, z_n) = 2^{d_{g,n}} \sum_{d_i} \prod_i d\hat{\xi}_{d_i}(z_i) \left\langle \prod_i \psi_i^{d_i} \prod_i f_{d_i}(\psi_i) e^{\frac{1}{2} \sum_{k,l} \hat{B}_{k,l} l_* \psi^k \psi'^l} e^{\sum_k \tilde{t}_k \kappa_k} \right\rangle_{g,n}. \quad (6.50)$$

Those changes of basis are very useful for the topological vertex below.

7 Topological vertex, proof of theorem 4.1

Here, we prove theorem 4.1 by applying theorem 3.3 to the topological vertex. This mostly consists in computing Laplace transforms.

Consider the topological vertex curve with framing f (see [1]). The 1-leg framed topological vertex's spectral [29] curve is $\mathcal{S}_{\text{vertex}} = (\mathbb{C} \setminus]-\infty, 0] \cup [1, +\infty[, x(z) = -f \ln z - \ln(1 - z), y(z) = -\ln z, B(z, z') = dz \otimes dz' / (z - z')^2)$, which satisfies:

$$e^{-x} = e^{-fy} (1 - e^{-y}). \quad (7.1)$$

It is most often written with the exponential \mathbb{C}^* variables $X = e^{-x}$ and $Y = e^{-y} = z$:

$$X = Y^f (1 - Y). \quad (7.2)$$

The only branchpoint is at $z = a = \frac{f}{f+1}$, at which we have

$$X(a) = e^{-x(a)} = \frac{f^f}{(f+1)^{f+1}}. \quad (7.3)$$

Just observe that changing $z \rightarrow 1/z$ is equivalent to changing $f \rightarrow -f - 1$ in $x(z)$, it changes $y(z) \rightarrow -y(z)$ and it doesn't change $B(z_1, z_2)$, and therefore all \tilde{t}_k and $\hat{B}_{k,l}$ are unchanged by changing $f \rightarrow -f - 1$:

$$\tilde{t}_k(-f - 1) = \tilde{t}_k(f) \quad , \quad \hat{B}_{k,l}(-f - 1) = \hat{B}_{k,l}(f). \quad (7.4)$$

Similarly, changing $z \rightarrow 1 - z$ and $x \rightarrow 1/fx$, is equivalent to changing $f \rightarrow 1/f$. $B(z_1, z_2)$ is unchanged, but in the expansion in powers of $x - x(a)$, the change $x \rightarrow x/f$ induces powers of f . This changes $\tilde{t}_k \rightarrow \tilde{t}_k f^{2k-1}$ and $\hat{B}_{k,l} \rightarrow \hat{B}_{k,l} f^{k+l+1}$:

$$\tilde{t}_k(1/f) = f^{2k-1} \tilde{t}_k(f) \quad , \quad \hat{B}_{k,l}(1/f) = f^{k+l+1} \hat{B}_{k,l}(f). \quad (7.5)$$

Those symmetry properties are of course the consequences of the fact that \mathbb{C}^3 is a toric Calabi-Yau 3-fold.

7.0.1 Computing \tilde{t}_k

If we assume $f \in \mathbb{R}_+$, we have $a = f/(1+f) \in]0, 1[$, and the steepest descent contour γ passing through the branchpoint, such that $x(\gamma) - x(a) = \mathbb{R}_+$, is simply

$$\gamma = [0, 1]. \quad (7.6)$$

The Laplace transform $e^{-g(u)}$ of ydx is easily written in terms of the variable z , using Eq. (6.14), and gives an Euler Beta-function:

$$\begin{aligned}
e^{-g(u)} &= \frac{2 u^{1/2} (f+1)^{(f+1)u}}{f^{fu} \sqrt{\pi}} \int_0^1 z^{fu} (1-z)^u dz/z \\
&= \frac{2 u^{1/2} (f+1)^{(f+1)u}}{f^{fu} \sqrt{\pi}} \frac{\Gamma(fu) \Gamma(1+u)}{\Gamma((f+1)u+1)} \\
&= \frac{2 u^{1/2} (f+1)^{(f+1)u}}{(f+1) f^{fu} \sqrt{\pi}} \frac{\Gamma(fu) \Gamma(u)}{\Gamma((f+1)u)}
\end{aligned} \tag{7.7}$$

Stirling's large u expansion of the Γ function gives (see appendix C)

$$\ln \Gamma(u) = u \ln u - u + \frac{1}{2} \ln(2\pi/u) + \sum_{k \geq 1} \frac{\mathcal{B}_{2k}}{2k(2k-1)} u^{1-2k} \tag{7.8}$$

where \mathcal{B}_k is the k^{th} Bernoulli number. That gives

$$e^{\tilde{t}_0} = \sqrt{\frac{f(f+1)}{8}} \tag{7.9}$$

and for $k \geq 1$, $\tilde{t}_{2k=0}$ and

$$\tilde{t}_{2k-1} = \frac{\mathcal{B}_{2k}}{2k(2k-1)} \left(\frac{1}{(f+1)^{2k-1}} - \frac{1}{f^{2k-1}} - 1 \right). \tag{7.10}$$

Notice that it indeed satisfies the symmetries Eq. (7.4) and Eq. (7.5).

7.0.2 Computing ξ_d

We have

$$d\xi_0(z) = - \operatorname{Res}_{z' \rightarrow a} B(z, z') \frac{1}{\sqrt{x(z') - x(a)}}, \tag{7.11}$$

or integrating once:

$$\xi_0(z) = \operatorname{Res}_{z' \rightarrow a} \frac{dz'}{z - z'} \frac{1}{\sqrt{x(z') - x(a)}}. \tag{7.12}$$

The pole is a simple pole and the residue is easily computed and gives

$$\xi_0(z) = \sqrt{\frac{2}{x''(a)}} \frac{1}{z - a} = \sqrt{\frac{2f}{(1+f)^3}} \frac{1}{z - \frac{f}{1+f}}. \tag{7.13}$$

Notice that $x'(z) = \frac{z(1+f)-f}{z(1-z)}$, and thus we can also write

$$\xi_0(z) = \sqrt{\frac{2f}{f+1}} \frac{1}{z(1-z)} \frac{dz}{dx(z)}. \tag{7.14}$$

Then, for $d \geq 1$, we have

$$\xi_d(z) = (2d-1)!! 2^{-d} \operatorname{Res}_{z' \rightarrow a} \frac{dz'}{z-z'} \frac{1}{(x(z')-x(a))^{d+1/2}}, \quad (7.15)$$

which shows that $\xi_d(z)$ must be a rational fraction of z , with a pole of degree $2d+1$ at $z=a$ and no other pole, and which must behave as:

$$\xi_d(z) \sim \frac{(2d-1)!! 2^{-d}}{(x(z)-x(a))^{(d+1/2)}} + O(1). \quad (7.16)$$

Since $x'(z)$ is a rational fraction:

$$x'(z) = \frac{z(1+f)-f}{z(1-z)}, \quad (7.17)$$

we see that $-d\xi_d(z)/dx(z)$ is also a rational fraction of z , and it clearly has a pole only at $z=a$, and near that pole, it behaves like (see Eq. (6.19))

$$-d\xi_d(z)/dx(z) \sim \frac{(2d+1)!! 2^{-d-1}}{(x(z)-x(a))^{(d+3/2)}} + \frac{(2d-1)!! 2^{-d-1} B_{2d,0}}{\sqrt{x(z)-x(a)}} + O(1), \quad (7.18)$$

which proves that

$$\xi_{d+1}(z) = -\frac{d\xi_d(z)}{dx(z)} - \hat{B}_{d,0} \xi_0(z), \quad (7.19)$$

and then

$$\xi_d = (-1)^d \xi_0^{(d)} - \sum_{k=0}^{d-1} (-1)^k \hat{B}_{d-1-k,0} \xi_0^{(k)} = -\sum_{k=0}^d (-1)^k \hat{B}_{d-1-k,0} \xi_0^{(k)}, \quad (7.20)$$

where we defined $\hat{B}_{-1,0} = -1$, and $\xi_0^{(d)} = (d/dx)^d \xi_0$. We thus have

$$\begin{aligned} & 2^{-dg,n} W_n^{(g)}(z_1, \dots, z_n) \\ = & \sum_{d_i} \prod_i d\xi_{d_i}(z_i) \left\langle \prod_i \psi_i^{d_i} e^{\frac{1}{2} \sum_{k,l} \hat{B}_{k,l} l_* \psi^k \psi'^l} e^{\sum_k \tilde{t}_k \kappa_k} \right\rangle_{g,n} \\ = & \sum_{d_i, d'_i} \prod_i (-1)^{d'_i} (-\hat{B}_{d_i-d'_i-1,0}) d\xi_0^{(d'_i)}(z_i) \left\langle \prod_i \psi_i^{d_i} e^{\frac{1}{2} \sum_{k,l} \hat{B}_{k,l} l_* \psi^k \psi'^l} e^{\sum_k \tilde{t}_k \kappa_k} \right\rangle_{g,n} \\ = & \sum_{d_i} \prod_i (-1)^{d_i} d\xi_0^{(d_i)}(z_i) \left\langle \prod_i \left(-\sum_{k \geq -1} \hat{B}_{k,0} \psi_i^{d_i+k+1} \right) e^{\frac{1}{2} \sum_{k,l} \hat{B}_{k,l} l_* \psi^k \psi'^l} e^{\sum_k \tilde{t}_k \kappa_k} \right\rangle_{g,n} \\ = & \sum_{d_i} \prod_i (-1)^{d_i} d\xi_0^{(d_i)}(z_i) \left\langle \prod_i \psi_i^{d_i} \prod_i \left(1 - \sum_{k \geq 0} \hat{B}_{k,0} \psi_i^{k+1} \right) e^{\frac{1}{2} \sum_{k,l} \hat{B}_{k,l} l_* \psi^k \psi'^l} e^{\sum_k \tilde{t}_k \kappa_k} \right\rangle_{g,n} \end{aligned} \quad (7.21)$$

7.0.3 Computation of $\hat{B}_{0,k}$

writing $\zeta(z) = \sqrt{x(z) - x(a)}$, we have

$$d\xi_0(z) = -\sqrt{\frac{2}{x''(a)}} \frac{dz}{(z-a)^2} = -\frac{d\zeta}{\zeta^2} - \sum_k B_{0,k} \zeta^k d\zeta. \quad (7.22)$$

Let us compute the Laplace transform:

$$\begin{aligned} \int_{\gamma} (d\xi_0(z) + \frac{d\zeta}{\zeta^2}) e^{-u(x(z)-x(a))} &= -\sum_k B_{0,2k} \int_{-\infty}^{\infty} \zeta^{2k} d\zeta e^{-u\zeta^2} \\ &= -\sum_k B_{0,2k} \sqrt{\pi} u^{-k-1/2} \frac{(2k-1)!!}{2^k} \\ &= -2\sqrt{\pi} u \sum_k \hat{B}_{0,k} u^{-k-1} \end{aligned} \quad (7.23)$$

Since $d\xi_0(z) + \frac{d\zeta}{\zeta^2}$ is analytical at $z = a$, we may slightly deform the contour, let us say, surrounding a in the upper half-plane. We have

$$\begin{aligned} \int_{\gamma} \frac{d\zeta}{\zeta^2} e^{-u(x(z)-x(a))} &= \int_{-\infty}^{\infty} \frac{d\zeta}{\zeta^2} e^{-u\zeta^2} \\ &= -\int_{-\infty}^{\infty} e^{-u\zeta^2} d\frac{1}{\zeta} \\ &= \int_{-\infty}^{\infty} \frac{1}{\zeta} de^{-u\zeta^2} \\ &= -2u \int_{-\infty}^{\infty} d\zeta e^{-u\zeta^2} \\ &= -2\sqrt{\pi} u \end{aligned} \quad (7.24)$$

and

$$\begin{aligned} \int_{\gamma} d\xi_0(z) e^{-u(x(z)-x(a))} &= e^{ux(a)} \sqrt{\frac{2}{x''(a)}} \int_0^1 e^{-ux(z)} d\frac{1}{z-a} \\ &= u e^{ux(a)} \sqrt{\frac{2f}{(f+1)^3}} \int_0^1 \frac{dx(z)}{z-a} e^{-ux(z)} \\ &= u e^{ux(a)} \sqrt{\frac{2f}{f+1}} \int_0^1 \frac{dz}{z(1-z)} e^{-ux(z)} \\ &= u e^{ux(a)} \sqrt{\frac{2f}{f+1}} \int_0^1 \frac{dz}{z(1-z)} z^{fu} (1-z)^u \\ &= u e^{ux(a)} \sqrt{\frac{2f}{f+1}} \frac{\Gamma(u) \Gamma(fu)}{\Gamma((f+1)u)} \end{aligned}$$

$$\begin{aligned}
&= u \frac{(f+1)^{(f+1)u}}{f^{fu}} \sqrt{\frac{2f}{f+1}} \frac{\Gamma(u) \Gamma(fu)}{\Gamma((f+1)u)} \\
&= 2\sqrt{\pi u} e^{\sum_k \frac{B_{2k}}{2k(2k-1)} u^{1-2k}} u^{1-2k} (1+f^{1-2k}-(f+1)^{1-2k}) \quad (7.25)
\end{aligned}$$

Eventually, we get that

$$\sum_k \hat{B}_{0,k} u^{-k-1} = 1 - e^{\sum_k \frac{B_{2k}}{2k(2k-1)} u^{1-2k} (1+f^{1-2k}-(f+1)^{1-2k})} = 1 - e^{-\sum_{k>0} \tilde{t}_k u^{-k}} = 1 - e^{-g(u)}. \quad (7.26)$$

where we have redefined $g(u)$ without the term \tilde{t}_0 .

According to Eq. (7.21), we thus have:

$$\begin{aligned}
&2^{-dg,n} e^{\tilde{t}_0 \chi_{g,n}} W_n^{(g)}(z_1, \dots, z_n) \\
&= \sum_{d_i} \prod_i (-1)^{d_i} d\xi_0^{(d_i)}(z_i) \left\langle \prod_i \psi_i^{d_i} e^{\frac{1}{2} \sum_{k,l} \hat{B}_{k,l} l_* \psi^k \psi^{l*}} e^{\sum_{k>0} \tilde{t}_k (\kappa_k - \sum_{i=1}^n \psi_i^k)} \right\rangle_{g,n}. \quad (7.27)
\end{aligned}$$

7.0.4 Computation of $\hat{B}_{k,l}$

Following Eq. (6.19), we write

$$\xi_0(z) = \frac{1}{\zeta} - \sum_k B_{0,k} \frac{\zeta^{k+1}}{k+1} \quad (7.28)$$

and thus

$$\begin{aligned}
\xi_0^{(j)}(z) &= \left(\frac{d}{2\zeta d\zeta} \right)^j \xi_0(z) \\
&= \frac{(-1)^j (2j-1)!!}{2^j \zeta^{2j+1}} - \sum_k B_{0,k} \frac{(k-1)(k-3) \dots (k-1-2j)}{2^j} \zeta^{k-2j+1} \quad (7.29)
\end{aligned}$$

and that implies by the recursion Eq. (7.19)

$$\begin{aligned}
\xi_d(z) &= \frac{(2d-1)!! 2^{-d}}{\zeta^{2d+1}} \\
&\quad - (-1)^d \sum_l B_{0,l+2d} \zeta^{l+1} (l+2d-1)(l+2d-3) \dots (l+3) 2^{-d} \\
&\quad - \sum_{k=0}^{d-1} \sum_l (-1)^k \hat{B}_{d-1-k,0} B_{0,l} \zeta^{l-2k+1} (l-1) \dots (l-2k+3) 2^{-k} \quad (7.30)
\end{aligned}$$

and comparing with Eq. (6.19)

$$\xi_d(z) = \frac{(2d-1)!! 2^{-d}}{(x(z) - x(a))^{d+\frac{1}{2}}} - (2d-1)!! 2^{-d} \sum_l B_{2d,l} \frac{(x(z) - x(a))^{\frac{l+1}{2}}}{l+1} \quad (7.31)$$

we get:

$$\begin{aligned}
(2d-1)!! B_{2d,l} &= (-1)^d B_{0,l+2d} (l+2d-1)(l+2d-3)\dots(l+1) \\
&\quad + \sum_{k=0}^{d-1} (-1)^k 2^{d-k} \hat{B}_{d-1-k,0} B_{0,l+2k} (l+2k-1)(l+2k-3)\dots(l+1)
\end{aligned} \tag{7.32}$$

and therefore

$$\hat{B}_{d,l} = (-1)^d \hat{B}_{0,d+l} + \sum_{k=0}^{d-1} (-1)^k \hat{B}_{d-1-k,0} \hat{B}_{0,l+k} \tag{7.33}$$

Let us define the generating functions

$$\sum_{k \geq -1} \hat{B}_{0,k} u^{-k-1} = 1 - \sum_{k \geq 0} \hat{B}_{0,k} u^{-k-1} = e^{-g(u)} \tag{7.34}$$

and we remind that we have found that $g(-u) = -g(u)$. We have

$$\begin{aligned}
\sum_{k \geq 0} \sum_{l \geq 0} \hat{B}_{k,l} u^{-k} v^{-l} &= \sum_{k \geq 0} \sum_{l \geq 0} \sum_{j=0}^k (-1)^j \hat{B}_{0,k-j-1} \hat{B}_{0,l+j} u^{-k} v^{-l} \\
&= \sum_{l \geq 0} \sum_{j \geq 0} \sum_{k \geq j} (-1)^j \hat{B}_{0,k-j-1} \hat{B}_{0,l+j} u^{-k} v^{-l} \\
&= \sum_{l \geq 0} \sum_{j \geq 0} \sum_{k \geq -1} (-1)^j \hat{B}_{0,k} \hat{B}_{0,l+j} u^{-k-1-j} v^{-l} \\
&= -e^{-g(u)} \sum_{l \geq 0} \sum_{j \geq 0} (-1)^j \hat{B}_{0,l+j} u^{-j} v^{-l} \\
&= -e^{-g(u)} \sum_{m \geq 0} \hat{B}_{0,m} v^{-m} \sum_{j=0}^m (-1)^j u^{-j} v^j \\
&= -e^{-g(u)} u v \sum_{m \geq 0} \hat{B}_{0,m} \frac{v^{-m-1} + (-1)^m u^{-m-1}}{u+v} \\
&= e^{-g(u)} u v \frac{e^{-g(v)} - e^{-g(-u)}}{u+v}.
\end{aligned} \tag{7.35}$$

Finally, the generating function of $\hat{B}_{k,l}$ is:

$$\sum_{k \geq 0} \sum_{l \geq 0} \hat{B}_{k,l} u^{-k} v^{-l} = u v \frac{e^{-g(u)} e^{-g(v)} - 1}{u+v}. \tag{7.36}$$

This shows that the Laplace transform Eq. (3.21) of B is

$$\hat{B}(u, v) = \frac{1}{2} \frac{e^{-g(u)} e^{-g(v)}}{1/u + 1/v} \tag{7.37}$$

7.0.5 Rewriting using intersection numbers identities

Now, let us rewrite $\sum_{k,l} \hat{B}_{k,l} \psi^k \psi'^l$ using lemma A.1 in appendix A. At each step we have to compute

$$\sum_{k,l} \hat{B}_{k,l} \left\langle \psi^k \psi'^l e^{\sum_k \tilde{t}_k \kappa_k} \Psi \right\rangle_{g,n+2} \quad (7.38)$$

where Ψ is some polynomial in ψ_1, \dots, ψ_n , in particular Ψ doesn't involve any κ class. We write

$$\begin{aligned} & \sum_{k,l} \hat{B}_{k,l} \psi^k \psi'^l \\ = & -e^{-g(1/\psi)} \frac{e^{g(1/\psi)} - e^{g(-1/\psi')}}{\psi + \psi'} \\ = & -e^{-g(1/\psi)} \sum_m \sum_{j_1, \dots, j_m} \frac{\tilde{t}_{2j_1+1} \dots \tilde{t}_{2j_m+1}}{m!} \sum_{k=0}^{m-1+2\sum j_i} (-1)^k \psi^{m-1-k+2\sum j_i} \psi'^k \end{aligned} \quad (7.39)$$

The first identity of Lemma A.1 allows to replace it by

$$- \sum_m \sum_{j_1, \dots, j_m} \frac{\tilde{t}_{2j_1+1} \dots \tilde{t}_{2j_m+1}}{m!} \sum_{k=0}^{m-1+2\sum j_i} (-1)^k \kappa_{m-2-k+2\sum j_i} \psi'^k \quad (7.40)$$

The derivative with respect to \tilde{t}_{2j+1} is

$$- \sum_m \sum_{j_1, \dots, j_{m-1}} \frac{\tilde{t}_{2j_1+1} \dots \tilde{t}_{2j_{m-1}+1}}{(m-1)!} \sum_{k=0}^{m+2j+2\sum j_i} (-1)^k \kappa_{m-2-k+2j+2\sum j_i} \psi'^k \quad (7.41)$$

and the second identity of Lemma A.1 allows to replace it by

$$- \sum_{k=0}^{2j} (-1)^k \psi^{2j-k} \psi'^k \quad (7.42)$$

I.e. we have

$$\sum_{k,l} \hat{B}_{k,l} \left\langle \psi^k \psi'^l e^{\sum_k \tilde{t}_k \kappa_k} \Psi \right\rangle_{g,n+2} = \sum_j \sum_k \tilde{t}_{2j+1} (-1)^k \left\langle \psi^{2j-k} \psi'^k e^{\sum_k \tilde{t}_k \kappa_k} \Psi \right\rangle_{g,n+2} \quad (7.43)$$

7.0.6 The Hodge class

We thus see that the spectral curve's class appearing in theorem 3.3, is the product of 3 classes:

$$\left\langle e^{\sum_{k>0} \tilde{t}_k (\kappa_k - \sum_i \psi_i^k)} e^{\frac{1}{2} \sum_{\delta} \sum_{k,l} \hat{B}_{k,l} l_{\delta*} \psi^k \psi'^l} \right\rangle_{g,n} = \langle \Lambda(1) \Lambda(f) \Lambda(-1-f) \rangle_{g,n} \quad (7.44)$$

where

$$\Lambda(f) = e^{-\sum_k \frac{1}{f^{2k-1}} \frac{B_{2k}}{2k(2k-1)}} \left(\kappa_{2k-1} - \sum_{i=1}^n \psi_i^{2k-1} + \frac{1}{2} \sum_{\delta} \sum_{l=0}^{2k-2} (-1)^l l_{\delta*} \psi^l \psi'^{2k-2-l} \right) \quad (7.45)$$

Using Mumford's formula [32], we recognize the Hodge class.

$$\Lambda(f) = \sum_k (-1)^k f^{-k} c_k(\mathbb{E}) = \text{Hodge class}. \quad (7.46)$$

Theorem 3.3, then says that, for the topological vertex's spectral curve, we have

$$\begin{aligned} & W_n^{(g)}(z_1, \dots, z_n) \\ &= 2^{d_{g,n}} e^{-\tilde{t}_0 \chi_{g,n}} \sum_{d_1, \dots, d_n} \prod_i (-1)^{d_i} d \xi_0^{(d_i)}(z_i) \langle \psi_1^{d_1} \dots \psi_n^{d_n} \Lambda(1) \Lambda(f) \Lambda(-f-1) \rangle_{g,n}. \end{aligned} \quad (7.47)$$

In other words, we have re-proved that the "remodelling the B-model" proposal of Bouchard-Klemm-Mariño-Pasquetti (BKMP conjecture [28, 7]) is valid for the topological vertex. This theorem was in fact already proved by Chen [9] and Zhou [37], using cut and join equations.

7.0.7 Laplace transform and Mariño–Vafa form

Let write $\xi_0(z)$ in Laplace transform

$$\xi_0(z) = \sum_{\mu=0}^{\infty} C_{\mu} e^{-\mu x(z)} = \sum_{\mu=0}^{\infty} C_{\mu} X(z)^{\mu}, \quad (7.48)$$

This is equivalent to a Taylor expansion near $z = 1$, in powers of $X(z) = e^{-x(z)} = z^f (1-z)$. We thus have

$$\begin{aligned} C_{\mu} &= \text{Res}_{z \rightarrow 1} \xi_0(z) X(z)^{-\mu} \frac{dX(z)}{X(z)} \\ &= \frac{\sqrt{2}}{\sqrt{f(f+1)}} \text{Res}_{z \rightarrow 1} \frac{1}{(f+1)z - f} X(z)^{-\mu} \left(\frac{-f dz}{z} + \frac{dz}{1-z} \right) \\ &= \frac{\sqrt{2}}{\sqrt{f(f+1)}} \text{Res}_{z \rightarrow 1} \frac{1}{(f+1)z - f} X(z)^{-\mu} \frac{(z - f(1-z))dz}{z(1-z)} \\ &= \frac{\sqrt{2}}{\sqrt{f(f+1)}} \text{Res}_{z \rightarrow 1} X(z)^{-\mu} \frac{dz}{z(1-z)} \\ &= \frac{\sqrt{2}}{\sqrt{f(f+1)}} \text{Res}_{z \rightarrow 1} \frac{1}{z^{\mu f} (1-z)^{\mu}} \frac{dz}{z(1-z)} \\ &= -\frac{\sqrt{2}}{\sqrt{f(f+1)}} \frac{\Gamma(1+\mu(f+1))}{\mu! \Gamma(1+\mu f)} \end{aligned}$$

$$= -\frac{\sqrt{2}(f+1)}{f\sqrt{f(f+1)}} \frac{\Gamma(\mu(f+1))}{\mu! \Gamma(\mu f)} \quad (7.49)$$

This implies

$$\xi_0(z) = -\frac{\sqrt{2}(f+1)}{f\sqrt{f(f+1)}} \sum_{\mu} e^{-\mu x(z)} \frac{\Gamma(\mu(f+1))}{\mu! \Gamma(\mu f)} \quad (7.50)$$

and taking derivatives:

$$d\xi_0^{(d)}(z) = -(-\mu)^{d+1} dx(z) \frac{\sqrt{2}(f+1)}{f\sqrt{f(f+1)}} \sum_{\mu} e^{-\mu x(z)} \frac{\Gamma(\mu(f+1))}{\mu! \Gamma(\mu f)}. \quad (7.51)$$

Then, write

$$\sum_{d_i} (-\mu_i)^{d_i+1} \psi_i^{d_i} = \frac{-\mu_i}{1 + \mu_i \psi_i} \quad (7.52)$$

That gives the Laplace transform of $W_n^{(g)}$ as:

$$\begin{aligned} W_n^{(g)}(z_1, \dots, z_n) &= 2^{d_{g,n}} e^{-\tilde{t}_0 \chi_{g,n}} (2(f+1)/f^3)^{n/2} \sum_{\mu_1, \dots, \mu_n} \\ &\prod_{i=1}^n \frac{\Gamma(\mu_i(f+1))}{\mu_i! \Gamma(\mu_i f)} \mu_i e^{-\mu_i x(z_i)} dx(z_i) \\ &\left\langle \prod_{i=1}^n \frac{1}{1 + \mu_i \psi_i} \Lambda(1) \Lambda(f) \Lambda(-f-1) \right\rangle_{g,n} \end{aligned} \quad (7.53)$$

which is the famous Mariño–Vafa formula [29, 25].

8 Examples

Let us show a few more examples.

8.1 Example: Weil-Petersson

Choose the Weil-Petersson curve:

$$y = \frac{1}{2\pi} \sin(2\pi\sqrt{x}) \quad (8.1)$$

or more precisely $\mathcal{S}_{\text{W.P.}} = (\mathbb{C}, x(z) = z^2, y(z) = \frac{1}{2\pi} \sin(2\pi z), B = \mathring{B})$. It has only one branchpoint at $z = a = 0$, and $B(z, z') = \mathring{B}(z, z') = dz \otimes dz' / (z - z')^2$.

We have

$$\begin{aligned} G(u) = e^{-g(u)} &= \frac{2u^{1/2}}{\sqrt{\pi}} \int e^{-ux} dy \\ &= \frac{2u^{1/2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-uz^2} \cos(2\pi z) dz \end{aligned}$$

$$\begin{aligned}
&= \frac{u^{1/2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-uz^2} (e^{2i\pi z} + e^{-2i\pi z}) dz \\
&= 2 \frac{u^{1/2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-uz^2} e^{2i\pi z} dz \\
&= 2 \frac{u^{1/2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u(z-i\pi/u)^2} e^{-\pi^2/u} dz \\
&= 2 e^{-\pi^2/u}
\end{aligned} \tag{8.2}$$

i.e.

$$g(u) = -\ln 2 + \pi^2/u \tag{8.3}$$

and thus

$$e^{\sum_k \tilde{t}_k \kappa_k} = 2^{-\kappa_0} e^{\pi^2 \kappa_1}. \tag{8.4}$$

We also have

$$\xi_d(z) = \frac{(2d-1)!!}{2^d z^{2d+1}}, \tag{8.5}$$

and thus

$$\begin{aligned}
&W_n^{(g)}(z_1, \dots, z_n) \\
&= (-1)^n 2^{d_{g,n} + \chi_{g,n}} \sum_{d_i} \prod_{i=1}^n \frac{(2d_i+1)!! dz_i}{2^{d_i} z_i^{2d_i+2}} \left\langle e^{\pi^2 \kappa_1} \prod_i \psi_i^{d_i} \right\rangle_{g,n} \\
&= (-1)^n 2^{d_{g,n} + \chi_{g,n}} \sum_{d_0+d_1+\dots+d_n=d_{g,n}} \prod_{i=1}^n \frac{(2d_i+1)!! dz_i}{2^{d_i} z_i^{2d_i+2}} \left\langle \frac{(\pi^2 \kappa_1)^{d_0}}{d_0!} \prod_{i=1}^n \psi_i^{d_i} \right\rangle_{g,n} \\
&= (-1)^n 2^{\chi_{g,n}} \sum_{d_0+d_1+\dots+d_n=d_{g,n}} \prod_{i=1}^n \frac{(2d_i+1)!! dz_i}{z_i^{2d_i+2}} \left\langle \frac{(2\pi^2 \kappa_1)^{d_0}}{d_0!} \prod_{i=1}^n \psi_i^{d_i} \right\rangle_{g,n} \\
&= (-1)^n 2^{\chi_{g,n}} \sum_{d_0+d_1+\dots+d_n=d_{g,n}} \prod_{i=1}^n \frac{(2d_i+1)! dz_i}{2^{d_i} d_i! z_i^{2d_i+2}} \left\langle \frac{(2\pi^2 \kappa_1)^{d_0}}{d_0!} \prod_{i=1}^n \psi_i^{d_i} \right\rangle_{g,n}
\end{aligned} \tag{8.6}$$

Notice that

$$\int_0^\infty L dL L^{2d} e^{-zL} = \frac{(2d+1)!}{z^{2d+2}} \tag{8.7}$$

therefore

$$\begin{aligned}
&\frac{W_n^{(g)}(z_1, \dots, z_n)}{dz_1 \dots dz_n} \\
&= (-1)^n 2^{\chi_{g,n}} \sum_{d_0+d_1+\dots+d_n=d_{g,n}} \prod_{i=1}^N \int_0^\infty L_i dL_i e^{-z_i L_i} \prod_{i=1}^n \frac{L_i^{2d_i}}{2^{d_i} d_i!} \left\langle \frac{(2\pi^2 \kappa_1)^{d_0}}{d_0!} \prod_{i=1}^n \psi_i^{d_i} \right\rangle_{g,n} \\
&= \frac{(-1)^n 2^{\chi_{g,n}}}{d_{g,n}!} \prod_{i=1}^N \int_0^\infty L_i dL_i e^{-z_i L_i} \left\langle (2\pi^2 \kappa_1 + \frac{1}{2} \sum_{i=1}^n L_i^2 \psi_i)^{d_{g,n}} \right\rangle_{g,n}
\end{aligned}$$

$$\begin{aligned}
&= (-1)^n 2^{\chi_{g,n}} \prod_{i=1}^N \int_0^\infty L_i dL_i e^{-z_i L_i} \left\langle e^{2\pi^2 \kappa_1 + \frac{1}{2} \sum_{i=1}^n L_i^2 \psi_i} \right\rangle_{g,n} \\
(8.8)
\end{aligned}$$

It is known (see Wolpert [34]) that $2\pi^2 \kappa_1$ is the Weil-Petersson form. In the Fenchel-Nielsen coordinates (l_i, θ_i) in Teichmüller space:

$$2\pi^2 \kappa_1 = \sum_i dl_i \wedge d\theta_i, \quad (8.9)$$

and thus, we have rederived that the symplectic invariants are the Laplace transform of the Weil Petersson volumes

$$\text{Vol}(L_1, \dots, L_n) = \left\langle e^{2\pi^2 \kappa_1 + \frac{1}{2} \sum_{i=1}^n L_i^2 \psi_i} \right\rangle_{g,n}. \quad (8.10)$$

The fact that symplectic invariants satisfy the topological recursion, is equivalent [31, 17, 26] (after Laplace transform), to the fact that Weil-Petersson volumes satisfy Mirzakhani's recursion relation [30].

8.2 Example: Lambert curve

Choose the Lambert curve $(\mathbb{C} \setminus \mathbb{R}_-, x(z) = -z + \ln z, y(z) = z, B = dz_1 \otimes dz_2 / (z_1 - z_2)^2)$, i.e. y as a function of e^x is the Lambert function:

$$e^x = ye^{-y} \quad \leftrightarrow \quad y = L(e^x). \quad (8.11)$$

We have

$$dx = \left(-1 + \frac{1}{z}\right) dz, \quad (8.12)$$

and thus there is a unique branchpoint (solution of $dx = 0$) at $a = 1, y = 1, x = -1$.

In principle, all the computations about the Lambert curve can be obtained by taking the $f \rightarrow \infty$ limit in the topological vertex [8, 29, 25], however, for completeness, let us rederive it directly.

8.2.1 The times \tilde{t}_k

The steepest descent path γ such that $x(\gamma) = [-1, +\infty[$, can be written in polar coordinates $z = \rho e^{i\theta}$, as $\rho = \theta / \sin \theta$, see fig. 1. It is easy to see that γ can be deformed into a contour surrounding the negative real axis \mathbb{R}_- .

We have:

$$e^{-g(u)} = \frac{2u^{1/2} e^{-u}}{\sqrt{\pi}} \int_{\gamma} (ye^{-y})^{-u} dy$$

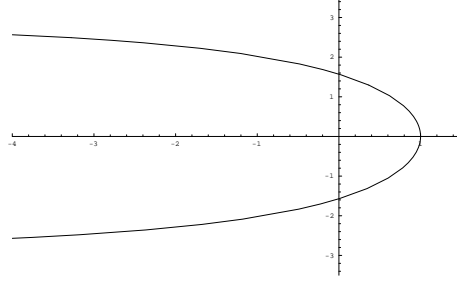


Figure 1: The steepest descent path for the Lambert curve. It surrounds the negative real axis. In polar coordinates, it has equation $\rho = \theta / \sin \theta$.

$$\begin{aligned}
&= \frac{2 u^{1/2} e^{-u}}{\sqrt{\pi}} \int_{\gamma} y^{-u} e^{uy} dy \\
&= 4i \sin \pi u \frac{u^{1/2} e^{-u}}{\sqrt{\pi}} \int_0^{\infty} y^{-u} e^{-uy} dy \\
&= 4i \sin \pi u \frac{u^{1/2} e^{-u}}{\sqrt{\pi}} u^{u-1} \Gamma(1-u) \\
&= 4i \sqrt{\pi} u^{-1/2} e^{-u} u^u \frac{1}{\Gamma(u)}
\end{aligned} \tag{8.13}$$

From the Stirling expansion:

$$\ln \Gamma(u) = u \ln u - u + \frac{1}{2} \ln(2\pi/u) + \sum_{k=1}^{\infty} \frac{\mathcal{B}_{2k}}{2k(2k-1)} u^{1-2k} \tag{8.14}$$

we find

$$\tilde{t}_0 = -\frac{1}{2} \ln 8 + \frac{i\pi}{2} \quad , \quad \tilde{t}_{2k-1} = \frac{\mathcal{B}_{2k}}{2k(2k-1)}. \tag{8.15}$$

We thus have to consider:

$$e^{\sum_{k \geq 1} \tilde{t}_k \kappa_k} = e^{-\sum_{k=1}^{\infty} \frac{\mathcal{B}_{2k}}{2k(2k-1)} \kappa_{2k-1}} = 1 - \frac{\kappa_1}{12} + \dots \tag{8.16}$$

8.2.2 Computing ξ_d

Like in section 7, we have

$$\xi_0(z) = \operatorname{Res}_{z' \rightarrow a} \frac{dz'}{z - z'} \frac{1}{\sqrt{x(z') - x(a)}} = \sqrt{\frac{2}{x''(a)}} \frac{1}{z - a} = \frac{-i\sqrt{2}}{z - 1}. \tag{8.17}$$

Notice that $x'(z) = \frac{1-z}{z}$, and thus we can also write

$$\xi_0(z) = \frac{i\sqrt{2}}{z} \frac{dz}{dx(z)}. \quad (8.18)$$

And like in section 7, since $x'(z) = \frac{1-z}{z}$ is a rational fraction with a zero only at $z = a = 1$, we see that $-d\xi_d(z)/dx(z)$ is also a rational fraction of z , and it clearly has a pole only at $z = a$, and near that pole, it behaves like (see Eq. (6.19))

$$-d\xi_d(z)/dx(z) \sim \frac{(2d+1)!! 2^{-d-1}}{(x(z) - x(a))^{(d+3/2)}} + \frac{(2d-1)!! 2^{-d-1} B_{2d,0}}{\sqrt{x(z) - x(a)}} + O(1), \quad (8.19)$$

which proves that

$$\xi_{d+1}(z) = -\frac{d\xi_d(z)}{dx(z)} - \hat{B}_{d,0} \xi_0(z), \quad (8.20)$$

and then

$$\xi_d = (-1)^d \xi_0^{(d)} - \sum_{k=0}^{d-1} (-1)^k \hat{B}_{d-1-k,0} \xi_0^{(k)} = -\sum_{k=0}^d (-1)^k \hat{B}_{d-1-k,0} \xi_0^{(k)}, \quad (8.21)$$

where we defined $\hat{B}_{-1,0} = -1$, and $\xi_0^{(d)} = (d/dx)^d \xi_0$. We thus have, like in section 7

$$\begin{aligned} & 2^{-d_{g,n}} W_n^{(g)}(z_1, \dots, z_n) \\ &= \sum_{d_i} \prod_i (-1)^{d_i} d\xi_0^{(d_i)}(z_i) \left\langle \prod_i \psi_i^{d_i} \prod_i (1 - \sum_{k \geq 0} \hat{B}_{k,0} \psi_i^{k+1}) e^{\frac{1}{2} \sum_{k,l} \hat{B}_{k,l} l_* \psi^k \psi'^l} e^{\sum_k \tilde{t}_k \kappa_k} \right\rangle_{g,n} \end{aligned} \quad (8.22)$$

8.2.3 Computation of $\hat{B}_{0,k}$

Like in section 7, we have

$$\int_{\gamma} (d\xi_0(z) + \frac{d\zeta}{\zeta^2}) e^{-u(x(z)-x(a))} = -2\sqrt{\pi u} \sum_k \hat{B}_{0,k} u^{-k-1}. \quad (8.23)$$

Since $d\xi_0(z) + \frac{d\zeta}{\zeta^2}$ is analytical at $z = a$, we may slightly deform the contour, let us say, surrounding a in the upper half-plane. We have

$$\int_{\gamma} \frac{d\zeta}{\zeta^2} e^{-u(x(z)-x(a))} = -2\sqrt{\pi u} \quad (8.24)$$

and

$$\begin{aligned} \int_{\gamma} d\xi_0(z) e^{-u(x(z)-x(a))} &= e^{ux(a)} \sqrt{\frac{2}{x''(a)}} \int_{\gamma} e^{-ux(z)} d\frac{1}{z-1} \\ &= u e^{-u} i \sqrt{2} \int_{\gamma} \frac{dx(z)}{z-1} e^{-ux(z)} \end{aligned}$$

$$\begin{aligned}
&= -u i \sqrt{2} e^{-u} \int_{\gamma} \frac{dz}{z} e^{-u x(z)} \\
&= -u i \sqrt{2} e^{-u} \int_{\gamma} \frac{dz}{z} e^{u z} z^{-u} \\
&= 2 \sin(\pi u) u i \sqrt{2} e^{-u} \int_0^{\infty} \frac{dz}{z} e^{-u z} z^{-u} \\
&= 2 \sin(\pi u) u i \sqrt{2} e^{-u} u^u \Gamma(-u) \\
&= 2\pi u i \sqrt{2} e^{-u} u^u \frac{1}{\Gamma(1+u)}
\end{aligned} \tag{8.25}$$

Eventually, we get that

$$\sum_k \hat{B}_{0,k} u^{-k-1} = 1 - e^{\sum_k \frac{B_{2k}}{2k(2k-1)} u^{1-2k}} = 1 - e^{-g(u)}. \tag{8.26}$$

where we have redefined $g(u)$ without the term \tilde{t}_0 .

We thus have:

$$\begin{aligned}
&2^{-d_{g,n}} e^{\tilde{t}_0 \chi_{g,n}} W_n^{(g)}(z_1, \dots, z_n) \\
&= \sum_{d_i} \prod_i (-1)^{d_i} d\xi_0^{(d_i)}(z_i) \left\langle \prod_i \psi_i^{d_i} e^{\frac{1}{2} \sum_{k,l} \hat{B}_{k,l} l_* \psi^k \psi'^l} e^{\sum_{k>0} \tilde{t}_k (\kappa_k - \sum_i \psi_i^k)} \right\rangle_{g,n}.
\end{aligned} \tag{8.27}$$

Then, all the same steps as in section 7 give that the generating function of $\hat{B}_{k,l}$ is:

$$\sum_{k \geq 0} \sum_{l \geq 0} \hat{B}_{k,l} u^{-k} v^{-l} = u v \frac{e^{-g(u)} e^{-g(v)} - 1}{u + v}. \tag{8.28}$$

And, using lemma A.1 as in section 7, we get

$$\sum_{k,l} \hat{B}_{k,l} \left\langle \psi^k \psi'^l e^{\sum_k \tilde{t}_k \kappa_k} \Psi \right\rangle_{g,n+2} = \sum_j \sum_k \tilde{t}_{2j+1} (-1)^k \left\langle \psi^{2j-k} \psi'^k e^{\sum_k \tilde{t}_k \kappa_k} \Psi \right\rangle_{g,n+2}. \tag{8.29}$$

8.2.4 The Hodge class

We thus see that the spectral curve's class appearing in theorem 3.3, is:

$$\left\langle e^{\sum_{k>0} \tilde{t}_k (\kappa_k - \sum_i \psi_i^k)} e^{\frac{1}{2} \sum_{\delta} \sum_{k,l} \hat{B}_{k,l} l_{\delta*} \psi^k \psi'^l} \right\rangle_{g,n} = \langle \Lambda(1) \rangle_{g,n} \tag{8.30}$$

where

$$\Lambda(1) = e^{-\sum_k \frac{B_{2k}}{2k(2k-1)} (\kappa_{2k-1} - \sum_{i=1}^n \psi_i^{2k-1} + \frac{1}{2} \sum_{\delta} \sum_{l=0}^{2k-2} (-1)^l l_{\delta*} \psi^l \psi'^{2k-2-l})} \tag{8.31}$$

i.e. using Mumford's formula [32], we recognize the Hodge class.

$$\Lambda(f) = \sum_k (-1)^k f^{-k} c_k(\mathbb{E}) = \text{Hodge class.} \quad (8.32)$$

Theorem 3.3, then says that, for the Lambert spectral curve, we have

$$\begin{aligned} & W_n^{(g)}(z_1, \dots, z_n) \\ &= 2^{d_{g,n}} e^{-\tilde{t}_0 \chi_{g,n}} \sum_{d_1, \dots, d_n} \prod_i (-1)^{d_i} d\xi_0^{(d_i)}(z_i) \langle \psi_1^{d_1} \dots \psi_n^{d_n} \Lambda(1) \rangle_{g,n}. \end{aligned} \quad (8.33)$$

In other words, we have re-proved the Bouchard-Mariño conjecture [8]. This theorem was in fact already proved in [6] using a matrix model, and in [15] using cut and join equations.

8.2.5 Laplace transform and ELSV form

Let us Laplace transform $\xi_0(z)$, i.e. expand it near $z = 0$, in powers of $X(z) = e^{x(z)} = z e^{-z}$:

$$\xi_0(z) = \sum_{\mu=0}^{\infty} C_{\mu} e^{\mu x(z)} = \sum_{\mu=0}^{\infty} C_{\mu} X(z)^{\mu}. \quad (8.34)$$

We have

$$\begin{aligned} C_{\mu} &= \text{Res}_{z \rightarrow 0} \xi_0(z) X(z)^{-\mu} \frac{dX(z)}{X(z)} \\ &= i\sqrt{2} \text{Res}_{z \rightarrow 0} \frac{1}{z} \frac{dz}{dx(z)} X(z)^{-\mu} dx(z) \\ &= i\sqrt{2} \text{Res}_{z \rightarrow 0} \frac{dz}{z} X(z)^{-\mu} \\ &= i\sqrt{2} \text{Res}_{z \rightarrow 0} \frac{dz}{z} z^{-\mu} e^{\mu z} \\ &= i\sqrt{2} \frac{\mu^{\mu}}{\mu!} \end{aligned} \quad (8.35)$$

This implies

$$\xi_0(z) = i\sqrt{2} \sum_{\mu} e^{\mu x(z)} \frac{\mu^{\mu}}{\mu!}. \quad (8.36)$$

and taking derivatives:

$$d\xi_0^{(d)}(z) = i\sqrt{2} \sum_{\mu} e^{\mu x(z)} \frac{\mu^{\mu}}{\mu!} \mu^{d+1} dx(z). \quad (8.37)$$

Then, write

$$\sum_{d_i} \mu_i^{d_i+1} \psi_i^{d_i} = \frac{\mu_i}{1 - \mu_i \psi_i} \quad (8.38)$$

That gives the Laplace transform of $W_n^{(g)}$ as:

$$W_n^{(g)}(z_1, \dots, z_n) = 2^{d_{g,n}} e^{-\tilde{t}_0 \chi_{g,n}} (-2)^{n/2} \sum_{\mu_1, \dots, \mu_n} \prod_{i=1}^n \frac{\mu_i^{\mu_i}}{\mu_i!} \mu_i e^{\mu_i x(z_i)} dx(z_i) \left\langle \prod_{i=1}^n \frac{1}{1 - \mu_i \psi_i} \Lambda(1) \right\rangle_{g,n}, \quad (8.39)$$

which is the famous ELSV formula [10, 11] for Hurwitz numbers.

8.3 Matrix models and Hankel class

Formal matrix model are generating functions enumerating discrete surfaces. Their correlation functions are defined as power series in t :

$$\omega_n^{(g)}(x_1, \dots, x_n; t; t_3, \dots, t_d) = \sum_{v=1}^{\infty} t^v \sum_{S \in \mathbb{M}_{g,n}(v)} \frac{1}{\#\text{Aut}(S)} \frac{t_3^{n_3(S)} t_4^{n_4(S)} \dots t_d^{n_d(S)}}{x_1^{1+l_1(S)} \dots x_n^{1+l_n(S)}} \quad (8.40)$$

where $\mathbb{M}_{g,n}(v)$ is the finite set of oriented discrete surfaces (also called "maps", see [4, 35]), made of polygonal faces of degree between 3 and d , of genus g , and with v vertices, and with n polygonal marked faces (and each marked face having one oriented marked edge). If $S \in \mathbb{M}_{g,n}(v)$, we call $n_j(S)$ the number of unmarked faces of degree j (and we have $j \geq 3$), we call $l_i(S)$ the degree of the i^{th} marked face, and $\#\text{Aut}(S)$ the cardinal of the automorphism group of S .

Most often, the dependence on $t; t_3, \dots, t_d$ will be implicitly understood, and we write

$$\omega_n^{(g)}(x_1, \dots, x_n) \equiv \omega_n^{(g)}(x_1, \dots, x_n; t; t_3, \dots, t_d). \quad (8.41)$$

It was proved in [12] that the generating functions $W_n^{(g)} = \omega_n^{(g)} dx_1 \dots dx_n$ satisfy the topological recursion, with a spectral curve given by [35]:

$$\mathcal{S}_{\text{Matrix}} = \begin{cases} \mathcal{C} = \mathbb{C}^* \\ x(z) = \alpha + \gamma(z + 1/z) \\ y(z) = \sum_{k=1}^{d-1} u_k z^{-k} \\ B(z_1, z_2) = \frac{dz_1 \otimes dz_2}{(z_1 - z_2)^2} \end{cases} \quad (8.42)$$

where the coefficients α, γ and u_k are determined by:

$$\begin{cases} \sum_k u_k (z^k + z^{-k}) = x(z) - \sum_{j=3}^d t_j x(z)^{j-1} \\ u_0 = 0 \\ u_1 = \frac{t}{\gamma} \end{cases} \quad (8.43)$$

and we choose the unique solution such that $\gamma^2 = t + O(t^2)$ and $\alpha = O(t)$.

• Example Quadrangulations

we choose $t_4 \neq 0$ and all other $t_j = 0$, that gives

$$\begin{cases} \gamma^2 = \frac{1-\sqrt{1-12tt_4}}{6t_4}, & \alpha = 0 \\ u_1 = \frac{t}{\gamma}, & u_2 = 0, \quad u_3 = -t_4\gamma^3 \end{cases} \quad (8.44)$$

and thus

$$\mathcal{S}_{\text{Quadrangulations}} = \begin{cases} \mathcal{C} = \mathbb{C}^* \\ x(z) = \gamma(z + 1/z) \\ y(z) = \frac{t}{\gamma z} - t_4\gamma^3 z^{-3} \\ B(z_1, z_2) = \frac{dz_1 \otimes dz_2}{(z_1 - z_2)^2} \end{cases} \quad (8.45)$$

Solving $x'(z) = 0$, we see that those spectral curves have 2 branchpoints, located at $z = a = \pm 1$. the case of multiple branchpoints will be done in a coming paper, but for the moment, let us compute the spectral curve class associated to the branch point at $a = 1$.

Assuming α and γ real positive, The steepest descent path going through $a = 1$, is simply $\gamma = [0, \infty[$. The Laplace transform gives

$$\begin{aligned} \int_{\gamma} e^{-u(x(z)-x(a))} dy(z) &= \gamma \int_0^{\infty} e^{-u\gamma(z+1/z-2)} \sum_k k u_k z^{-k} \frac{dz}{z} \\ &= \gamma e^{2u\gamma} \sum_k k u_k \int_{-\infty}^{\infty} e^{-2u\gamma \cosh \phi} e^{-k\phi} d\phi \\ &= \pi \gamma e^{2u\gamma} \sum_k k u_k H_k(2i\gamma u) \end{aligned} \quad (8.46)$$

where H_k is the k^{th} Hankel function of the 1st kind (which is closely related to the Bessel function). Therefore

$$e^{-\sum_k \tilde{t}_k u^{-k}} = 2\gamma \sqrt{\pi u} e^{2u\gamma} \sum_k k u_k H_k(2i\gamma u) \quad (8.47)$$

We also have

$$\xi_0(z) = \sqrt{\frac{2}{x''(a)}} \frac{1}{z-a} = \frac{1}{\sqrt{\gamma}} \frac{1}{z-1} \quad (8.48)$$

and

$$\xi_d(z) = \frac{(2d-1)!!}{2^d \gamma^{d+1/2}} \sum_{k=0}^{2d} \frac{1}{(z-1)^{2d+1-k}} \binom{d+1/2}{k}. \quad (8.49)$$

This computation can be in principle pursued, and would give the number of quadrangulations (or other discrete surfaces) in terms of intersection numbers. This will be the purpose of another work.

8.3.1 Example: resolved conifold

One can also try to apply the general formula to the Resolved conifold's spectral curve, in order to check the BKMP conjecture.

The conifold's spectral curve \mathcal{S} is $\mathcal{S} = (\mathbb{C}, x(z) = -f \ln z + \ln(1 - z) - \ln(1 - qz), y(z) = -\ln z, B(z_1, z_2) = dz_1 dz_2 / (z_1 - z_2)^2)$, it satisfies

$$e^{-x} = e^{-fy} \frac{1 - e^{-y}}{1 - q e^{-y}}. \quad (8.50)$$

It is most often written with the exponential variables $X = e^{-x}$ and $Y = e^{-y} = z$, as:

$$X = Y^f \frac{1 - Y}{1 - qY}. \quad (8.51)$$

There are 2 branchpoints, $a_+ > 0$ and $a_- < \ln q$. We assume $0 < q < 1$, and thus the steepest descent paths for the Laplace transforms are $z \in \gamma_+ = [0, 1]$, and $z \in \gamma_- =]1/q, \infty[$.

The Laplace transforms $e^{-g_{\pm}(u)}$ of ydx are easily written in terms of the variable z , and give hypergeometric functions of q :

$$e^{-g_{\pm}(u)} = \frac{2u^{1/2} e^{u a_{\pm}}}{\sqrt{\pi}} \int_{\gamma_{\pm}} z^{fu} (1 - z)^u (1 - qz)^{-u} dz/z \quad (8.52)$$

Thus

$$\begin{aligned} e^{-g_+(u)} &= \frac{2u^{1/2} e^{u a_+}}{\sqrt{\pi}} \int_0^1 z^{fu} (1 - z)^u (1 - qz)^{-u} dz/z \\ &= \frac{2u^{1/2} e^{u a_+}}{\sqrt{\pi}} \frac{\Gamma(fu)\Gamma(u+1)}{\Gamma(fu+u+1)} {}_2F_1(u, fu; fu+u+1; q) \end{aligned} \quad (8.53)$$

and, by a simple change of variable $z \rightarrow q/z$:

$$\begin{aligned} e^{-g_-(u)} &= \frac{2u^{1/2} e^{u a_-}}{\sqrt{\pi}} \int_{1/q}^{\infty} z^{fu} (1 - z)^u (1 - qz)^{-u} dz/z \\ &= e^{-g_+(-u)}. \end{aligned} \quad (8.54)$$

However, it is not so simple to compute explicitly the large u expansion of $g_{\pm}(u)$, and this computation will be pursued in other works.

9 Conclusion

We have found the interpretation of symplectic invariants of a spectral curve, in terms of integrals over the moduli-space $\mathcal{M}_{g,n}$.

With this formula, we have found new proofs of the Bouchard-Mariño conjecture [8] for Hurwitz numbers, and BKMP conjecture [7] for \mathbb{C}^3 . We hope that the extension of the formula for several branchpoints, could help prove the BKMP conjecture for more complicated toric geometries, but there is still some substantial work ahead.

Remarks about Mirror symmetry

- Intersection numbers "count" complex curves with marked points, in some moduli-space of curves. They are related to a type A topological string theory. The moduli which appear in the intersection numbers are the \tilde{t}_k and $\hat{B}_{k,l}$'s and $d\xi_d(z)$.

- On the other hand, symplectic invariants are defined in terms of moduli of the spectral curve, and in particular in terms of the Bergman kernel $B(z_1, z_2)$ and in terms of the 1-form ydx . They are obtained by computing residues, i.e. in terms of the complex geometry on the spectral curve. They can be thought of as a type B topological string theory.

We see that the relationship between the type A moduli and the type B moduli, is the Laplace transform, for instance:

$$e^{-\sum_k \tilde{t}_k u^{-k}} = \frac{2 u^{3/2} e^{ux(a)}}{\sqrt{\pi}} \int_{\gamma_a} e^{-ux} y dx \quad (9.1)$$

relates the moduli \tilde{t}_k of κ -classes to the 1-form ydx . The moduli of ψ classes, encoded in $d\xi_d$ and in $\hat{B}_{k,l}$, are related to the Laplace transform of the Bergman kernel.

Notice also that the steepest descent contour γ_a , defined as

$$\text{Im } x(\gamma) = \text{constant} \quad (9.2)$$

or equivalently

$$\text{Arg}(X(\gamma)) = \text{Arg}(e^{-x(\gamma)}) = \text{constant} \quad (9.3)$$

is closely related to the definition of Lagrangian submanifolds. Indeed, write

$$X = |X| e^{-\theta}, \quad (9.4)$$

the steepest descent contour γ_a is a contour along which $d\theta = 0$.

We thus see that there seems to be a deep link between this computation, and mirror symmetry, but this link is still to be clarified.

Namely, it seems important to understand how the Laplace transform of ydx is related to κ -classes, and the Laplace transform of B is related to ψ -classes !

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Appendix A

A Some relationships among intersection numbers

Lemma A.1 *We have the following identities for intersection numbers*

$$\left\langle \kappa_d \prod_{i=1}^n \psi_i^{d_i} \right\rangle_{g,n} = \left\langle \psi_{n+1}^{d+1} e^{-\sum_j \tilde{t}_j \psi_{n+1}^j} e^{\sum_k \tilde{t}_k \kappa_k} \prod_{i=1}^n \psi_i^{d_i} \right\rangle_{g,n+1}, \quad (\text{A.1})$$

$$\left\langle \psi_{n+1}^{d+1} e^{\sum_k \tilde{t}_k \kappa_k} \prod_{i=1}^n \psi_i^{d_i} \right\rangle_{g,n+1} = \sum_m \sum_{j_1, \dots, j_m} \frac{\tilde{t}_{j_1} \cdots \tilde{t}_{j_m}}{m!} \left\langle \kappa_{d+\sum j_i} e^{\sum_k \tilde{t}_k \kappa_k} \prod_{i=1}^n \psi_i^{d_i} \right\rangle_{g,n}. \quad (\text{A.2})$$

proof:

Those identities can be deduced from direct geometric properties of tautological classes, similar to [3].

However, let us show a proof based on general properties of symplectic invariants specialized to theorem 3.2.

Let us consider an infinitesimal variation of spectral curve:

$$y \rightarrow y + \delta y \quad (\text{A.3})$$

in other words, since $y(z) = \sum_k t_{k+2} z^k$:

$$t_k \rightarrow t_k + \delta t_k. \quad (\text{A.4})$$

This induces a variation of the times \tilde{t}_k through Laplace transform:

$$\delta(e^{-g(u)}) = -\delta g(u) e^{-g(u)} = \frac{2u^{3/2}}{\sqrt{\pi}} \int e^{-ux} \delta y \, dx. \quad (\text{A.5})$$

- Let us consider a function $\delta y(z) = -y^{(-d)}(z)$ such that:

$$\left(\frac{d}{dx(z)}\right)^d \delta y(z) = -y(z) \quad , \quad x(z) = z^2, \quad (\text{A.6})$$

i.e. more explicitly

$$y^{(-d)}(z) = \sum_k t_{k+2} \frac{2^d}{(k+2)(k+4)\dots(k+2d)} z^{k+2d}. \quad (\text{A.7})$$

By integration by parts we compute δg :

$$\delta g(u) e^{-g(u)} = u^{-d} \frac{2u^{3/2}}{\sqrt{\pi}} \int e^{-ux} y \, dx = u^{-d} e^{-g(u)} \quad (\text{A.8})$$

i.e.

$$\delta g(u) = u^{-d}, \quad (\text{A.9})$$

i.e.

$$\tilde{t}_k \rightarrow \tilde{t}_k + \delta_{k,d} \delta \tilde{t}_d, \quad (\text{A.10})$$

i.e. our infinitesimal variation δ is in fact

$$\delta = \frac{\partial}{\partial \tilde{t}_d}. \quad (\text{A.11})$$

On the other hand, we compute the dual cycle to the variation δy of the spectral curve as (form-cycle duality is realized by the Bergman kernel):

$$\delta y(z) \, dx(z) = \text{Res}_{z' \rightarrow \infty} B(z, z') y^{(-d-1)}(z') \quad (\text{A.12})$$

and the special geometry property of symplectic invariants then implies that:

$$\delta W_n^{(g)}(z_1, \dots, z_n) = \text{Res}_{z' \rightarrow \infty} W_{n+1}^{(g)}(z_1, \dots, z_n, z') y^{(-d-1)}(z'), \quad (\text{A.13})$$

and since the only poles of $W_{n+1}^{(g)}$ are at the branchpoint $z = 0$, we may move the integration contour and get:

$$\delta W_n^{(g)}(z_1, \dots, z_n) = - \text{Res}_{z' \rightarrow 0} W_{n+1}^{(g)}(z_1, \dots, z_n, z') y^{(-d-1)}(z'). \quad (\text{A.14})$$

From theorem 3.2, we thus have

$$\begin{aligned} & (-1)^n 2^{3g-3+n} \frac{\partial}{\partial \tilde{t}_d} \left\langle e^{\sum_k \tilde{t}_k \kappa_k} \prod_{i=1}^n \psi_i^{d_i} \right\rangle_{g,n} \\ = & -(-1)^{n+1} 2^{3g-3+n+1} \text{Res}_{z' \rightarrow 0} y^{(-d-1)}(z') \sum_{d'} \frac{(2d'+1)!! \, dz'}{2^{d'} z'^{2d'+2}} \left\langle \psi_{n+1}^{d'} e^{\sum_k \tilde{t}_k \kappa_k} \prod_{i=1}^n \psi_i^{d_i} \right\rangle_{g,n+1} \end{aligned}$$

(A.15)

using Eq. (A.7), the residues give:

$$\begin{aligned} \left\langle \kappa_d e^{\sum_k \tilde{t}_k \kappa_k} \prod_{i=1}^n \psi_i^{d_i} \right\rangle_{g,n} &= \sum_k \frac{t_{2k+3} (2k+1)!!}{2^k} \left\langle \psi_{n+1}^{k+d+1} e^{\sum_k \tilde{t}_k \kappa_k} \prod_{i=1}^n \psi_i^{d_i} \right\rangle_{g,n+1} \\ &= \left\langle \psi_{n+1}^{d+1} e^{-\sum_k \tilde{t}_k \psi_{n+1}^k} e^{\sum_k \tilde{t}_k \kappa_k} \prod_{i=1}^n \psi_i^{d_i} \right\rangle_{g,n+1}. \end{aligned} \quad (\text{A.16})$$

This ends the proof of the first identity. Notice that when all $\tilde{t}_k = 0$, this identity is well known [3].

• Now let us prove the other identity. We choose

$$\delta y(z) = z^{2d+1}. \quad (\text{A.17})$$

That gives

$$-\delta g(u) e^{-g(u)} = \frac{2u^{3/2}}{\sqrt{\pi}} \int e^{-ux} x^{d+1/2} dx = \frac{(2d+1)!!}{2^{d-1} u^d} \quad (\text{A.18})$$

i.e.

$$\begin{aligned} \delta g(u) &= -\frac{(2d+1)!!}{2^{d-1} u^d} e^{g(u)} \\ &= -\frac{(2d+1)!!}{2^{d-1}} \sum_m \sum_{j_1, \dots, j_m} \frac{\tilde{t}_{j_1} \dots \tilde{t}_{j_m}}{m!} u^{\sum j_i - d} \end{aligned} \quad (\text{A.19})$$

which implies

$$\delta e^{\sum_k \tilde{t}_k \kappa_k} = -\frac{(2d+1)!!}{2^{d-1}} \sum_m \sum_{j_1, \dots, j_m} \frac{\tilde{t}_{j_1} \dots \tilde{t}_{j_m}}{m!} \kappa_{\sum j_i - d} e^{\sum_k \tilde{t}_k \kappa_k}, \quad (\text{A.20})$$

and thus

$$\begin{aligned} &\delta W_n^{(g)}(z_1, \dots, z_n) \\ &= -(-1)^n 2^{3g-3+n} \frac{(2d+1)!!}{2^{d-1}} \sum_m \sum_{j_1, \dots, j_m} \frac{\tilde{t}_{j_1} \dots \tilde{t}_{j_m}}{m!} \sum_{d_1, \dots, d_n} \\ &\quad \prod_{i=1}^n \frac{(2d_i+1)!! dz_i}{2^{d_i} z_i^{2d_i+2}} \left\langle \kappa_{\sum j_i - d} e^{\sum_k \tilde{t}_k \kappa_k} \prod_i \psi_i^{d_i} \right\rangle_{g,n}. \end{aligned} \quad (\text{A.21})$$

On the other hand, we can compute $\delta W_n^{(g)}$ from the special geometry property. The dual of δy is given by:

$$\delta y(z) dx(z) = \frac{-2}{2d+3} \operatorname{Res}_{z' \rightarrow \infty} B(z, z') z'^{2d+3} \quad (\text{A.22})$$

and thus

$$\begin{aligned}
& \delta W_n^{(g)}(z_1, \dots, z_n) \\
&= \frac{-2}{2d+3} \operatorname{Res}_{z' \rightarrow \infty} W_{n+1}^{(g)}(z_1, \dots, z_n, z') z'^{2d+3} \\
&= (-1)^{n+1} 2^{3g-3+n+1} \frac{2(2d+3)!!}{(2d+3)2^{d+1}} \sum_{d_1, \dots, d_n} \prod_{i=1}^n \frac{(2d_i+1)!! dz_i}{2^{d_i} z_i^{2d_i+2}} \left\langle e^{\sum_k \tilde{t}_k \kappa_k} \prod_{i=1}^n \psi_i^{d_i} \psi_{n+1}^{d+1} \right\rangle_{g, n+1}.
\end{aligned} \tag{A.23}$$

Comparing those two expressions of $\delta W_n^{(g)}$ completes the proof.

□

B Table of intersection numbers

We organize them by Euler characteristics.

$-\chi$	$d_{g,n} \rightarrow 0$	1	2	3	4
1	$\langle 1 \rangle_{0,3} = 1$	$\langle \tau_1 \rangle_{1,1} = \frac{1}{24}$ $\langle \kappa_1 \rangle_{1,1} = \frac{1}{24}$			
2		$\langle \tau_1 \rangle_{0,4} = 1$ $\langle \kappa_1 \rangle_{0,4} = 1$	$\langle \tau_2 \rangle_{1,2} = \frac{1}{24}$ $\langle \tau_1^2 \rangle_{1,2} = \frac{1}{24}$ $\langle \tau_1 \kappa_1 \rangle_{1,2} = \frac{1}{12}$ $\langle \kappa_2 \rangle_{1,2} = \frac{1}{24}$ $\langle \kappa_1^2 \rangle_{1,2} = \frac{1}{8}$	$\langle \kappa_3 \rangle_{2,0} = \frac{1}{3^2 2^6}$ $\langle \kappa_2 \kappa_1 \rangle_{2,0} = \frac{1}{5 \cdot 3 \cdot 2^3}$ $\langle \kappa_1^3 \rangle_{2,0} = \frac{43}{5 \cdot 3^2 \cdot 2^5}$	
3			$\langle \tau_2 \rangle_{0,5} = 1$ $\langle \tau_1^2 \rangle_{0,5} = 2$ $\langle \tau_1 \kappa_1 \rangle_{0,5} = 3$ $\langle \kappa_2 \rangle_{0,5} = 1$ $\langle \kappa_1^2 \rangle_{0,5} = 5$	$\langle \tau_3 \rangle_{1,3} = \frac{1}{24}$ $\langle \tau_2 \tau_1 \rangle_{1,3} = \frac{1}{12}$ $\langle \tau_1^3 \rangle_{1,3} = \frac{1}{12}$ $\langle \tau_2 \kappa_1 \rangle_{1,3} = \frac{1}{6}$ $\langle \tau_1^2 \kappa_1 \rangle_{1,3} = \frac{1}{4}$ $\langle \tau_1 \kappa_2 \rangle_{1,3} = \frac{1}{8}$ $\langle \tau_1 \kappa_1^2 \rangle_{1,3} = \frac{13}{24}$ $\langle \kappa_3 \rangle_{1,3} = \frac{1}{24}$ $\langle \kappa_2 \kappa_1 \rangle_{1,3} = \frac{1}{4}$ $\langle \kappa_1^3 \rangle_{1,3} = \frac{7}{36}$	$\langle \tau_4 \rangle_{2,1} = \frac{1}{3^2 2^7}$ $\langle \tau_3 \kappa_1 \rangle_{2,1} = \frac{29}{5 \cdot 3^2 \cdot 2^7}$ $\langle \tau_2 \kappa_2 \rangle_{2,1} = \frac{29}{5 \cdot 3^2 \cdot 2^7}$ $\langle \tau_2 \kappa_1^2 \rangle_{2,1} = \frac{139}{5 \cdot 3^2 \cdot 2^7}$ $\langle \tau_1 \kappa_3 \rangle_{2,1} = \frac{1}{3 \cdot 2^7}$ $\langle \tau_1 \kappa_2 \kappa_1 \rangle_{2,1} = \frac{101}{5 \cdot 3^2 \cdot 2^7}$ $\langle \tau_1 \kappa_1^3 \rangle_{2,1} = \frac{169}{5 \cdot 3 \cdot 2^7}$ $\langle \kappa_4 \rangle_{2,1} = \frac{1}{3^2 2^7}$ $\langle \kappa_3 \kappa_1 \rangle_{2,1} = \frac{39}{5 \cdot 3^2 \cdot 2^7}$ $\langle \kappa_2^2 \rangle_{2,1} = \frac{53}{5 \cdot 3^2 \cdot 2^7}$ $\langle \kappa_2 \kappa_1^2 \rangle_{2,1} = \frac{777}{5 \cdot 3^3 \cdot 2^7}$ $\langle \kappa_1^4 \rangle_{2,1} = \frac{29}{2^7}$

A few easy general relations are

$$\langle \tau_1^{n-3} \rangle_{0,n} = (n-3)! \tag{B.1}$$

$$\langle \tau_1^k \Psi \rangle_{0,n} = \frac{(n-3)!}{(n-3-k)!} \langle \Psi \rangle_{n-k} \tag{B.2}$$

$$\langle \tau_1^{n-5} \tau_2 \rangle_{0,n} = \frac{(n-3)!}{2}, \quad \langle \tau_1^{n-6} \tau_3 \rangle_{0,n} = \frac{(n-3)!}{3!} \tag{B.3}$$

$$< \tau_1^{n-7} \tau_2^2 >_{0,n} = \frac{(n-3)!}{4!} 6 \quad (\text{B.4})$$

$$< \tau_{3g-2} >_{g,1} = < \kappa_{3g-3} >_{g,0} = \frac{1}{24^g g!} \quad (\text{B.5})$$

C Stirling approximation

We have

$$\Gamma(u) = \int_0^\infty dz z^{u-1} e^{-z} dz \quad (\text{C.1})$$

And it has the large u asymptotic expansion

$$\ln \Gamma(u) = u \ln u - u + \frac{1}{2} \ln(2\pi/u) + \sum_{k=1}^\infty \frac{B_{2k}}{2k(2k-1)} \frac{1}{u^{2k-1}} \quad (\text{C.2})$$

where B_k is the k^{th} Bernoulli number:

$$B_2 = \frac{1}{6}, \quad B_4 = \frac{-1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = \frac{-1}{30}, \quad B_{10} = \frac{5}{66}, \quad B_{12} = \frac{-691}{2730}, \dots \quad (\text{C.3})$$

The Euler Beta function is:

$$B(u, v) = \int_0^1 dz z^{u-1} (1-z)^{v-1} = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}. \quad (\text{C.4})$$

D Proof of Lemma 6.1

We prove it by recursion on $2g - 2 + n$.

We shall always use the local parameter $z = \zeta = \sqrt{x(z) - x(a)}$. We have, in the small z expansion:

$$B(z_1, z_2) = \frac{dz_1 \otimes dz_2}{(z_1 - z_2)^2} + \sum_{k,l} B_{k,l} z_1^k z_2^l dz_1 \otimes dz_2. \quad (\text{D.1})$$

First, notice that the recursive definition of $W_n^{(g)}$ involves computing $2g - 2 + n$ residues each containing a Bergman kernel, and also some residues may involve one or two $W_2^{(0)} = B$. Eventually, we see that $W_n^{(g)}$ is a polynomial in the $B_{k,l}$'s of degree at most $d_{g,n} = 3g - 3 + n$, and also, since we compute residues at each step, Taylor series near $z = 0$ can be truncated to the order of poles, and this means that each $W_n^{(g)}$ involves only a finite number of $B_{k,l}$'s.

Therefore, there is no loss of generality in assuming that only a finite number of $B_{k,l}$'s are non-vanishing. Let us also assume for the moment that $B_{k,l}$ and $B_{l,k}$ are independent variables, but in the end we will have to choose $B_{k,l} = B_{l,k}$.

Our goal is to prove by recursion that:

$$\begin{aligned} \left(\frac{\partial}{\partial B_{l,k}} + \frac{\partial}{\partial B_{k,l}} \right) W_n^{(g)}(J) &= \text{Res}_{z \rightarrow \infty} \text{Res}_{z' \rightarrow \infty} \frac{z^{k+1}}{k+1} \frac{z'^{l+1}}{l+1} \left[W_{n+2}^{(g-1)}(z, z', J) \right. \\ &\quad \left. + \sum_h \sum_{I \subset J} W_{1+\#I}^{(h)}(z, I) W_{1+n-\#I}^{(g-h)}(z', J \setminus I) \right]. \end{aligned} \quad (\text{D.2})$$

Initialization of the recursion

Notice that

$$- \text{Res}_{z' \rightarrow \infty} B(z, z') \frac{z'^{k+1}}{k+1} = z^k dz \quad (\text{D.3})$$

Therefore we have

$$\begin{aligned} \frac{\partial B(z_1, z_2)}{\partial B_{k,l}} &= z_1^k dz_1 \otimes z_2^l dz_2 \\ &= \text{Res}_{z \rightarrow \infty} \text{Res}_{z' \rightarrow \infty} \frac{z^{k+1}}{k+1} B(z, z_1) \frac{z'^{l+1}}{l+1} B(z', z_2). \end{aligned} \quad (\text{D.4})$$

This is the initial case $2g - 2 + n = 0$ for the recursion:

$$\begin{aligned} &\left(\frac{\partial}{\partial B_{k,l}} + \frac{\partial}{\partial B_{l,k}} \right) W_2^{(0)}(z_1, z_2) \\ &= \text{Res}_{z \rightarrow \infty} \text{Res}_{z' \rightarrow \infty} \frac{z^{k+1}}{k+1} \frac{z'^{l+1}}{l+1} \left[W_2^{(0)}(z, z_1) W_2^{(0)}(z', z_2) + W_2^{(0)}(z, z_2) W_2^{(0)}(z', z_1) \right]. \end{aligned} \quad (\text{D.5})$$

This implies for the recursion kernel $K(z_0, z)$ defined in Eq. (??):

$$\frac{\partial K(z_0, z_1)}{\partial B_{k,l}} = \text{Res}_{z \rightarrow \infty} \text{Res}_{z' \rightarrow \infty} \frac{z^{k+1}}{k+1} \frac{z'^{l+1}}{l+1} B(z_0, z) K(z', z_1). \quad (\text{D.6})$$

Assume that we have proved the lemma for every $2g' - 2 + n' < 2g - 2 + n$. We have (where $J = \{z_1, \dots, z_n\}$):

$$W_{n+1}^{(g)}(z_0, J) = \text{Res}_{z'' \rightarrow 0} K(z_0, z'') \left[W_{n+2}^{(g-1)}(z'', -z'', J) + \sum_h \sum_{I \subset J} W_{1+\#I}^{(h)}(z'', I) W_{1+n-\#I}^{(g-h)}(-z'', J \setminus I) \right] \quad (\text{D.7})$$

and thus

$$\begin{aligned} &\frac{\partial}{\partial B_{k,l}} W_{n+1}^{(g)}(z_0, J) \\ &= \text{Res}_{z'' \rightarrow 0} \frac{\partial}{\partial B_{k,l}} K(z_0, z'') \left[W_{n+2}^{(g-1)}(z'', -z'', J) \right. \\ &\quad \left. + \sum_h \sum_{I \subset J} W_{1+\#I}^{(h)}(z'', I) W_{1+n-\#I}^{(g-h)}(-z'', J \setminus I) \right] \end{aligned}$$

$$\begin{aligned}
& + \operatorname{Res}_{z'' \rightarrow 0} K(z_0, z'') \left[\frac{\partial}{\partial B_{k,l}} W_{n+2}^{(g-1)}(z'', -z'', J) \right. \\
& + \sum_h \sum_{I \subset J} \frac{\partial}{\partial B_{k,l}} W_{1+\#I}^{(h)}(z'', I) W_{1+n-\#I}^{(g-h)}(-z'', J \setminus I) \\
& \left. + W_{1+\#I}^{(h)}(z'', I) \frac{\partial}{\partial B_{k,l}} W_{1+n-\#I}^{(g-h)}(-z'', J \setminus I) \right] \\
& \quad (D.8)
\end{aligned}$$

The first term, with $\partial K(z_0, z'')/\partial B_{k,l}$ gives simply

$$\operatorname{Res}_{z \rightarrow \infty} \operatorname{Res}_{z' \rightarrow \infty} \frac{z^{k+1}}{k+1} \frac{z'^{l+1}}{l+1} B(z_0, z) W_{n+1}^{(g)}(z', J) \quad (D.9)$$

Let us now focus on the second term, i.e. $\partial/\partial B_{k,l} + \partial/\partial B_{l,k}$ of the bracket. From the recursion hypothesis, it gives

$$\begin{aligned}
(1) \quad & \left| \operatorname{Res}_{z \rightarrow \infty} \operatorname{Res}_{z' \rightarrow \infty} \frac{z^{k+1}}{k+1} \frac{z'^{l+1}}{l+1} \left[W_{n+4}^{(g-2)}(z'', -z'', z, z', J) \right. \right. \\
(2) \quad & \left| + \sum_{h', I' \subset J} W_{3+\#I'}^{(h')}(z'', -z'', z, I') W_{1+n-\#I'}^{(g-1-h')}(z', J \setminus I') \right. \\
(3) \quad & \left| + \sum_{h', I' \subset J} W_{2+\#I'}^{(h')}(z'', z, I') W_{2+n-\#I'}^{(g-1-h')}(-z'', z', J \setminus I') \right. \\
(4) \quad & \left| + \sum_{h', I' \subset J} W_{2+\#I'}^{(h')}(-z'', z, I') W_{2+n-\#I'}^{(g-1-h')}(z'', z', J \setminus I') \right. \\
(5) \quad & \left| + \sum_{h', I' \subset J} W_{1+\#I'}^{(h')}(z, I') W_{3+n-\#I'}^{(g-1-h')}(z'', -z'', z', J \setminus I') \right. \\
(6) \quad & \left| + \sum_{h, I \subset J} W_{3+\#I}^{(h-1)}(z'', z, z', I) W_{1+n-\#I}^{(g-h)}(-z'', J \setminus I) \right. \\
(7) \quad & \left| + \sum_{h, I \subset J} \sum_{h', I' \subset I} W_{2+\#I'}^{(h')}(z'', z, I') W_{1+\#I-\#I'}^{(h-h')}(z', I \setminus I') W_{1+n-\#I}^{(g-h)}(-z'', J \setminus I) \right. \\
(8) \quad & \left| + \sum_{h, I \subset J} \sum_{h', I' \subset I} W_{1+\#I'}^{(h')}(z, I') W_{2+\#I-\#I'}^{(h-h')}(z'', z', I \setminus I') W_{1+n-\#I}^{(g-h)}(-z'', J \setminus I) \right. \\
(9) \quad & \left| + \sum_{h, I \subset J} W_{1+\#I}^{(h)}(z'', I) W_{3+n-\#I}^{(g-h-1)}(z, z', -z'', J \setminus I) \right. \\
(10) \quad & \left| + \sum_{h, I \subset J} \sum_{h', I' \subset J \setminus I} W_{1+\#I}^{(h)}(z'', I) W_{2+\#I'}^{(h')}(z, -z'', I') W_{1+n-\#I-\#I'}^{(g-h-h')}(z', (J \setminus I) \setminus I') \right. \\
(11) \quad & \left| + \sum_{h, I \subset J} \sum_{h', I' \subset J \setminus I} W_{1+\#I}^{(h)}(z'', I) W_{1+\#I'}^{(h')}(z, I') W_{2+n-\#I-\#I'}^{(g-h-h')}(-z'', z', (J \setminus I) \setminus I') \right] \\
& \quad (D.10)
\end{aligned}$$

Now, we multiply by $K(z_0, z'')$ and take the residue at $z'' \rightarrow 0$, then, by definition of $W_n^{(g)}$'s, terms (2) + (7) + (10) give

$$\sum_{h', I' \subset J} W_{2+\#I'}^{(h')}(z_0, z, I') W_{1+n-\#I'}^{(g-h')}(z', J \setminus I'), \quad (D.11)$$

terms (5) + (8) + (11) give

$$\sum_{h', I' \subset J} W_{1+\#I'}^{(h')}(z, I') W_{2+n-\#I'}^{(g-h')}(z_0, z', J \setminus I'), \quad (\text{D.12})$$

and terms (1) + (3) + (4) + (6) + (9) give

$$W_{3+n}^{(g-1)}(z_0, z, z', J). \quad (\text{D.13})$$

And thus finally:

$$\begin{aligned} & \left(\frac{\partial}{\partial B_{k,l}} + \frac{\partial}{\partial B_{l,k}} \right) W_{n+1}^{(g)}(z_0, J) \\ = & \text{Res}_{z \rightarrow \infty} \text{Res}_{z' \rightarrow \infty} \frac{z^{k+1}}{k+1} \frac{z'^{l+1}}{l+1} \left[W_{3+n}^{(g-1)}(z_0, z, z', J) \right. \\ & + B(z_0, z) W_{n+1}^{(g)}(z', J) + W_{n+1}^{(g)}(z, J) B(z_0, z') \\ & + \sum_{h', I' \subset J} W_{2+\#I'}^{(h')}(z_0, z, I') W_{1+n-\#I'}^{(g-h')}(z', J \setminus I') \\ & \left. + \sum_{h', I' \subset J} W_{1+\#I'}^{(h')}(z, I') W_{2+n-\#I'}^{(g-h')}(z_0, z', J \setminus I') \right] \end{aligned} \quad (\text{D.14})$$

which proves our recursion hypothesis to order $2g - 2 + n + 1$.

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